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H^∞-Optimal control for a class of reaction–diffusion equations

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This paper investigates the control of laminar combustion processes represented by a class of one-dimensional, non-linear, reaction–diffusion equations. The control objective is to regulate the spatial location of a propagating flame front dynamically by varying the fuel–air velocity. It is shown that when viewed from an input–output perspective, the problem is equivalent to the control of a system consisting of linear elements and a single time-varying non-linear element. The system is controlled using a linear compensator that is designed based on an H^∞ (minimax) optimality criterion.

1. Introduction

The desire to improve the performance of gas turbine engines has naturally led to a need to develop procedures for sensing, controlling, and optimizing engine operating characteristics. Of particular interest is the optimization of the combustion processes occurring in the engine's combustor. The goals of such an optimization might be to minimize heat flux to critical areas of the combustor, control operation near stability limits, minimize undesirable by-products, and maximize the efficiency and operating range of the engine. At present, the approach to this optimization problem is generally static in nature and usually involves solving for the geometry of the combustor. It is of practical interest to consider the use of an active control system to regulate the combustion process in the engine dynamically in order to modify and improve its performance.

A control system for a combustion process might be designed to regulate such quantities as temperature profile, pressure distribution, and species concentrations in the flow field. The control system would necessarily have available a set of variables that can be manipulated to alter the combustor's properties. These might include fuel concentration, inlet conditions, and combustor geometry. In order to study the design of control systems for these processes, it is necessary to examine the nature and the state-of-the-art of models for combustion processes. The complexity and accuracy of these models will have a strong bearing on determining whether certain theoretical and analytical approaches are practical or intractable.

In general, models for combustors are described by non-linear, mixed hyperbolic–parabolic partial differential equations for such properties as velocity and pressure distributions, temperature profile, and species concentrations. The optimal control of systems described by partial differential equations is a problem in the general area of distributed parameter control. In practice, combustion models cannot accurately predict the behaviour of an actual system due to uncertainty in such physical parameters as the reaction rates of the various species, the boundary

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conditions, and the influence of turbulence and its inherently statistical nature. In addition, the non-linearity and extremely high—in theory infinite—order of these models precludes the use of conventional design techniques. Consequently, it is desirable and necessary to investigate control strategies and design procedures that do not rely heavily on the use of an accurate model.

Previous work on the control of combustion processes has involved the application of simple or heuristic strategies to one-dimensional laminar combustion equations (Dzielski and Nagurka 1985, Ramos 1984, 1986). The heuristic control law proposed in Ramos (1986) results in a system which exhibits very lightly damped non-linear oscillations. A very similar heuristic control policy proposed in Ramos (1984) has been found to have a response time several orders of magnitude ($\sim 10^3$) slower than that achieved by a proportional-plus-integral control law (Dzielski and Nagurka 1985). In addition, the highly oscillatory behaviour of the responses achieved in Ramos (1984, 1986) suggests that the desired equilibrium state is weakly stable. With the exception of Dzielski and Nagurka (1985), previous work has not included any theoretical justification and has relied solely on the use of simulations of a single equation describing the temperature profile to establish results; therefore, the validity of these results is intimately tied to the particular model equation in which all quantities are assumed known. As a consequence, the results do not provide a basis for a practical approach for the control of combustion processes.

In view of the preliminary nature of previous investigations, research has been aimed at broadening the applicability of the results. The goal of this work has been to develop an approach which requires a minimum amount of quantitative information from a dynamical model—or perhaps only qualitative information from the physics—to design compensators that can be rigorously justified. This paper is organized as follows. In § 2, a general class of one-dimensional combustion problems is described. Viewed from an input–output perspective, this class of problems can be shown to reduce to a single linear element and a single time-varying non-linear element. In § 3, the desired control objectives are related to constraints on the sensitivity function of an approximation of the system. Owing to the presence of uncertainty in the model, as well as uncertainty in the non-linearity, it is necessary to choose a linear compensator that guarantees the stability of the actual system. The search for an optimal trade-off between the competing performance requirements and a stability constraint leads to a frequency domain optimization of the minimax or H^∞ type. In § 4, an application of this method to a simple example is presented.

2. Input–output model for a class of reaction–diffusion equations

The class of reaction–diffusion equations that has been investigated can be characterized by the dimensionless partial differential equation (Williams 1965)

$$T_t + VT_x = T_{xx} + f(T) \tag{1}$$

where $x \in [L^-, L^+]$ is the spatial variable, T denotes temperature with boundary conditions $T(L^-) = 0$, $T(L^+) = 1$, t denotes a temporal variable, and $f(T) \in C^1[0, 1]$ is an algebraic function of the temperature satisfying

$$f(0) = f(1) = 0 \quad \text{and} \quad \frac{df(0)}{dx} > 0 \tag{2}$$

with a first derivative that changes sign once in $[0, 1]$. Equation (1) describes the time evolution of the temperature distribution, $T(x, t)$, in a moving, one-dimensional, laminar, reacting medium. The function $f(T)$ is called the reaction rate function. For some choices of $f(T)$ and for $x \in (-\infty, +\infty)$ there exist explicit solutions to (1). However, it will be assumed from the outset that the spatial domain is bounded since (i) physically we tend to think of the fuel–air mixture as being injected at some finite cold boundary of the system, and (ii) it must be truncated for numerical solution. Since boundary effects have not been considered in deriving (1), this equation does not accurately describe the process if the flame front approaches too near either boundary. The parameter V in (1) is related to the flow rate of the medium and will subsequently be referred to as the fuel–air velocity. Subject to the specified boundary conditions, the solution to (1) is a monotonically increasing function in the spatial variable as illustrated in Fig. 1. The wavefront propagates either to the left or right depending on the value of V.

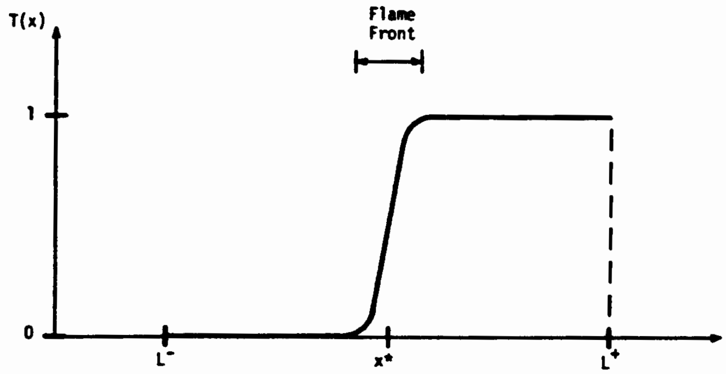

Figure 1. General shape of the solution of the class of systems described by reaction–diffusion Equation (1).

The control objective for this class of problems is to find a strategy for adjusting the fuel–air velocity, V, such that a well-defined point on the temperature profile is driven to a desired location in the spatial domain, $x^* = x_{\mathrm{desired}}$. This point is defined by a function of the form $F[T(x^*, t)] = 0$ and could be the inflection point of the profile, $F[T] = T_{xx}(x^*, t) = 0$, or perhaps the point where the temperature takes on a certain value, $F[T] = T(x^*, t) - T_0 = 0$. The only requirements are that the point exists and is unique at all times, and that it is possible to make measurements that can be related to its location, x^*. In physical terms, the problem is one of regulating the location of the combustion front as defined by the quantity x^*.

In general, it is not possible to measure directly the quantity x^*. Its value is determined from sensor measurements of the temperature profile, $T(x, t)$. These measurements, y_i, are of the form

$$y_i = g_i(T(x, t)); \quad i = 1, \ldots, N \tag{3}$$

where g_i is the influence function of the sensor on the medium, and N is the number of sensors. Naturally, for these measurements to be useful, there must be some functional relationship between the sensor outputs and the measured location of the point x^*, denoted x^*_{m};

$$x^*_{\mathrm{m}} = h(y_1, \ldots, y_N) \tag{4}$$

A distinction has been made between the physical quantity, x^*, and the measured quantity, x_m^*, since a precise calculation of x^* is based on exact knowledge of the current temperature profile, and the profile cannot be determined from a finite number of measurements. The form of the function h depends strongly on the number of sensors. In principle, a continuous measurement of the temperature profile would permit an exact determination of x^*. On the other hand, for the case of a single measurement, it is not possible to determine accurately the location of the flame front, except when the front is located in the immediate vicinity of the (single) sensor. From the definition of x^*, the measurement function implicitly inverts the temperature profile, and as can be seen from Fig. 1, this inverse is nearly singular except in the neighbourhood of the front. In addition, due to the assumed uncertainty in the function $f(T)$ in (1), h cannot be determined accurately.

Given a suitably defined reference, x_{desired}, an error is defined as

$$e = x_m^* - x_{\text{desired}} = h(y_1, \ldots, y_N) - x_{\text{desired}} \tag{5}$$

and the compensation objective is to regulate the error by adjusting the fuel–air velocity, V.

Consider the dynamical relation between V and x^* implied by (1). Since the rate at which the wavefront of the temperature profile propagates into the medium is related to V, the relationship between V and x^* can be expressed functionally as

$$\frac{dx^*}{dt} = \phi_1(V, T(x, t)) \tag{6}$$

where ϕ_1 denotes the functional relationship. There exists a non-zero value of V for which $dx^*/dt = 0$ and the shape of the temperature profile does not change with time. This value of V is known as the steady-state wave speed and is denoted by V^*. It is related to the function $f(T)$ by

$$V^* = \int_{-\infty}^{+\infty} f(T_{ss}(x))\, dx \tag{7}$$

where $T_{ss}(x)$ is the steady-state solution corresponding to the eigenvalue V^* and $x \in (-\infty, +\infty)$. Therefore, uncertainty in the reaction rate function can be interpreted as uncertainty in the value of V^* as well as uncertainty in the solution $T(x, t)$. This suggests that it may be better to write (6) as

$$\begin{aligned}\frac{dx^*}{dt} &= \phi(V - V^*, T(x, t)) \\ &= \phi(V_i, T(x, t))\end{aligned} \tag{8}$$

where ϕ denotes the functional relationship and V_i denotes an input variable which when equal to zero causes the flame front to remain stationary. Naturally, the lack of knowledge about the function f leads to uncertainty in the functional value of ϕ for given values of the arguments. It should be noted that ϕ is a non-linear time-varying function. The class of systems described by (1) can be represented by the block diagram shown in Fig. 2.

In Fig. 2 it is assumed that the fuel–air velocity, V, is manipulated by the controller and that the steady-state wave speed, V^*, is a constant. Equation (7) suggests that it is also possible to assume that the fuel–air velocity, V, is constant and to use the reaction rate function, $f(T)$, as the control variable. Manipulating the shape of the reaction rate function corresponds to altering the composition of the fuel–air mixture. This approach is in a sense dual to the one considered here since the roles of V and V^* in Fig. 2 are interchanged. The former problem in which V is the control variable is addressed here because the relationship between physical quantities and the reaction rate functions is not as well understood.

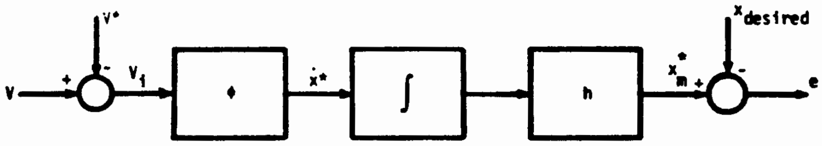

Figure 2. Input–output block diagram representing the class of systems described by (1) under assumptions.

No additional simplifying assumptions have been made beyond those made in the derivation of (1). The dynamics of the system described by (1) are implicit in the relations h and ϕ. The difference between the two characterizations is only in point of view, i.e., a state-space description (1) vs. an input–output description (3), (8). The advantage of the input–output representation is that the functional relationships between the various quantities are displayed clearly. The complexity of partial differential equations of the form (1) often hides relationships among variables. It will be seen in the following section that these relationships, which are often overlooked, can be very simple; not exploiting them can lead to acceptance of undesirable results. In this paper, the determination of ϕ and h has been based on numerical considerations, although in practice it could easily be based on experimental results.

3. Compensation problem

This section addresses the problem of designing linear compensators for the class of plants characterized by (1). The designs are based on the input–output model of Fig. 2. The problem is to find a linear compensator to correct the error, $e(t)$, by manipulating the controlled variable, V, such that the closed-loop system has desirable stability and performance characteristics. An optimality criterion is developed for choosing compensators for the closed-loop system.

Although it will be shown that ϕ can be approximated by a unity gain, for practical reasons (to be discussed) it will be represented by a linear dynamical system. Following this, conditions will be placed on the resulting loop such that the closed-loop system will reject a specific set of disturbances. Additional restrictions will be placed on the transfer function consisting of the compensator and the linear approximation of ϕ to guarantee that power levels in the loop are reasonable. Since these considerations are based on a linear approximation, a method of guaranteeing the closed-loop stability of the non-linear system will be proposed. These discussions will lead to requirements on the shape of the system's sensitivity function or its complement in certain frequency ranges.

In the discussion, the H^∞ norm of a transfer function is used as a measure of system performance. The H^∞ norm of a complex function, $g(s)$, analytic in the closed right-half-plane is defined to be

$$\|g(s)\|_\infty = \operatorname*{ess\,sup}_{\omega \in R} |g(j\omega)| \tag{9}$$

where $|\cdot|$ denotes the usual complex modulus. This norm has been proposed for feedback controller design (Zames 1981) and for application to robustness optimization (Zames 1983). Here, each of the control objectives will be formulated as a desire to minimize a suitably weighted transfer function. The optimization of a weighted sum of these individual measures in the H^{∞} norm will yield the controller.

The direct linearization of an equation such as (1) for the relationship between V and x^* is a difficult task. In addition, linearization assumes significant knowledge of the model and relevant data, which, as mentioned earlier, is generally inaccurate. Numerical studies of (1) have been conducted to investigate the qualitative input–output behaviour of these systems. The results suggest that the relationship between V and dx^*/dt defined by (1) can be accurately represented by a unity gain. A physical justification for such a relationship can be obtained by examining the system modelled by (1). The flame is assumed to propagate relative to the medium at a constant rate determined by the reaction rate function, $f(T)$. The fuel–air velocity characterizes the flow rate of the medium and is the control variable. Therefore, changes in the flow rate of the medium cause an equal change in the propagation rate of the flame front. In actuality, the shape of the profile changes as V changes; however, the approximation is accurate due to the fact that the shape changes are very localized in the region of the flame front. The reaction rate functions which have been studied in detail are

$$f_1(T) = T^n(1 - T) \qquad (10)$$

for $n = 1, 2$. The class of functions (10) has been proposed as representative of flames having an excess of one reactant (Spalding 1957).

Equation (1) was simulated for $f(T)$ given by (10) with $n = 2$ using a time linearization algorithm. (This is the identical equation studied in Ramos 1986, and the reader should consult that paper and its references for a more detailed discussion of this equation and the numerical method.) The simulated responses of x^* to steps in V of varying magnitude are shown in Fig. 3. The point x^* is assumed to be the spatial

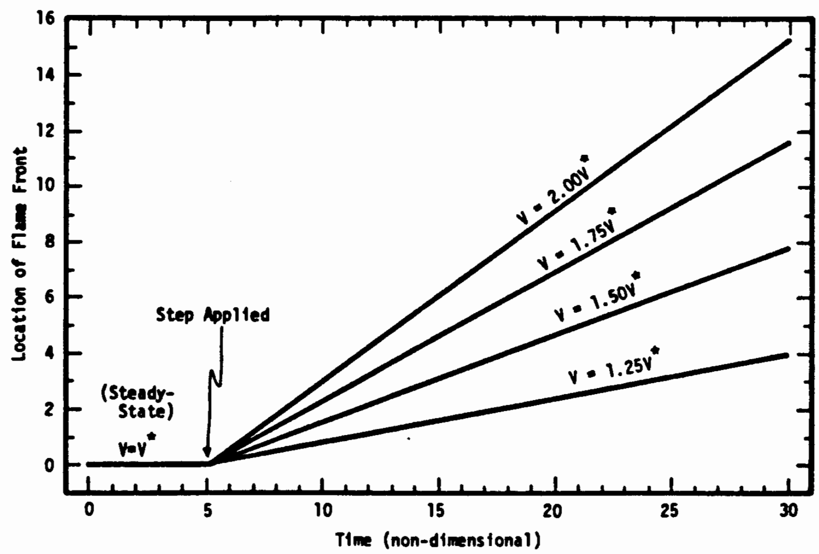

Figure 3. Response of (1) to steps in the fuel–air velocity.

location of the point $T(x^*) = 0{\cdot}5$. The simulations assume initial steady-state conditions, $V(0) = V^*$, with steps at increments of $V^*/4$. The responses for equal negative increments in the control are symmetrical with respect to the time axis.

The simulations show that there are no apparent transient dynamics associated with the steps and that the rate of propagation is proportional to the magnitude of the steps. Based on this observation, it seems that there is no need to resort to a heuristic approach as suggested in Ramos (1986) and that the problem can be treated as a classical servo problem with a single non-linearity, h. There are, however, physical effects that occur that are not modelled by (1) which should be considered in an overall design. These effects are particularly important for the dual problem of controlling with the reaction rate function $f(T)$. In this case, it becomes necessary to include the effects of a fuel transport delay, which can be approximated as

$$p(s) = \frac{k \exp(-\tau s)}{s + a} \tag{11}$$

The term $\exp(-\tau s)$ could represent a delay associated with fuel transport and the lag term, $s + a$, could account for the fact that neither the composition of the medium nor the fuel–air velocity can be changed instantaneously. We assume below that the operator ϕ has been modelled by a linear dynamical operator such as (11). It should be noted that the H^∞-optimization problem to be formulated here has not yet been solved for non-rational transfer functions and delays must also be approximated (progress is being made: Flamm and Mitter 1986).

The non-linear function h cannot be approximated accurately, and the only assumptions on its form are that it satisfies a sector criterion to be defined. The system for which a linear compensator, $c(s)$, is to be designed is illustrated in Fig. 4. This figure shows that the closed-loop system has been reduced to a set of linear components and a single time-varying non-linearity. In compensating the closed-loop system for disturbance rejection and power limitation, it will be assumed that the function h can be approximated by its tangent at the desired equilibrium state and that this approximation is incorporated into the approximation, $p(s)$, given by (11). The linear system, $p(s)/s$ with $h = 1$, will be regarded as the nominal system for which compensation is to be designed. Note that unless $f(T)$ is given explicitly, it is not possible to determine h or its tangent exactly.

The sensitivity function of a linear system is defined to be the transfer function from a signal entering additively at the plant output to the signal measured at the output of the system. For the nominal system of Fig. 4, the input signal enters

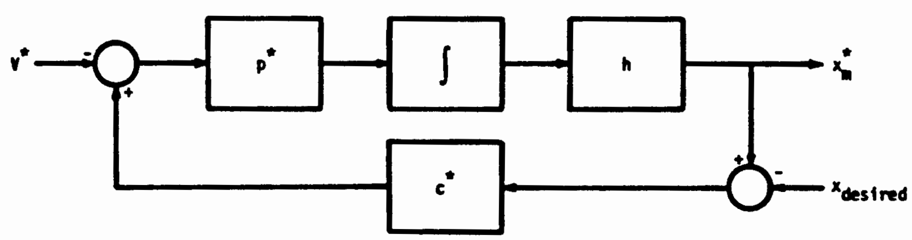

Figure 4. Feedback block diagram of closed-loop system (p^* and c^* denote convolution operators).

immediately following h and the output is x_m^*; therefore, the sensitivity function, $S(s)$, is

$$S(s) = \frac{1}{1 + \dfrac{c(s)p(s)}{s}} \tag{12}$$

The complementary sensitivity function, $T(s) = 1 - S(s)$, represents the transfer function from the reference signal, x_{desired}, to the plant output. In the following paragraphs, the desired performance characteristics of the system will be related to the values of the sensitivity function and its complement in certain frequency ranges.

In order for the nominal closed-loop system to reject constant disturbances at the plant input, such as the uncertain value of V^*, the final value theorem requires that

$$\lim_{s \to 0} s \frac{\dfrac{p(s)}{s}}{1 + \dfrac{c(s)p(s)}{s}} \frac{V^*}{s} = \lim_{s \to 0} S(s) \frac{p(s)}{s} V^* = 0 \tag{13}$$

where $S(s)$ is the system's sensitivity function. Since $p(s)$ is assumed to have no finite zeros, (13) implies that $S(s)$ must have a zero of order 2 at $s = 0$. This can be accomplished using a weighting $1/s^2$ on the sensitivity function and requiring that

$$\left\| \frac{1}{s^2} S(s) \right\|_\infty$$

be bounded. From the definition of the H^∞ norm, it is clear that for the above norm to be bounded on the imaginary axis, $S(s)$ must have a zero of at least the required order.

In addition, it is also desirable for the system to attenuate disturbances in V^* at the plant input. These disturbances could be attributed to environmental effects or variations due to transport. For attenuation of these disturbances reflected to the plant output, it is sufficient that the system sensitivity function, $S(s)$, be small in the frequency range in which the disturbance has its energy. If $W_d(s)$ denotes a weighting function reflecting the frequency content of the disturbance in the fuel–air velocity, acceptable disturbance rejection can be achieved by designing $S(s)$ such that

$$\left\| W_d(s) \frac{p(s)}{s} S(s) \right\|_\infty$$

is small. Combining this expression and the condition for rejection of constant disturbances yields an objective

$$\| J_1 \|_\infty = \left\| \frac{W_d(s)p(s)}{s^2} S(s) \right\|_\infty = \| W_1(s)S(s) \|_\infty \tag{14}$$

for minimization.

The model that has been postulated is assumed from the outset to include parameters that are simultaneously uncertain and time-varying. This requires a much stronger stability requirement than those typically used in robust compensation problems where the uncertainty is generally assumed constant. The fact that the closed-loop system of Fig. 4 can be separated into a linear block and a time-varying non-linear block suggests the use of one of the various stability criteria that have been developed for this class of problems (e.g. the circle criterion, Taylor 1980). Since many of these criteria are stated in the frequency domain in terms of the linear subsystem's transfer function, it is possible to relate these constraints on the open-loop to the system's sensitivity function and its complement. The circle criteria is employed here due to the fact that there is an interesting geometrical interpretation of the constraint that it imposes.

The circle criterion can be applied to find constants k_1 and k_2 such that $k_1 > 0$ and

$$k_1 \leqslant \frac{h(x^*, t)}{x^*} \leqslant k_2 \tag{15}$$

implies stability of the closed-loop system. The constants can be found from any circle drawn on the Nyquist diagram of the linear subsystem such that the circle does not intersect the Nyquist curve. The minimum and maximum of the negative inverse of the points where the circle intersects the real axis give k_1 and k_2, respectively.

Based on the monotonicity of the profile in Fig. 1, h will lie in the first and third quadrants of the $(x^*, h(x^*, t))$-plane. Note that the perturbations in the temperature profile caused by turbulence can be accommodated by designing the sensor influence functions and h such that the product $(h(x^*)/x^*)$ is strictly positive for the expected disturbances.

For a circle centred at $(-1, 0)$ in the complex plane, the radius of the largest circle that can be drawn without intersecting the Nyquist curve is

$$r = \inf \left| 1 + \frac{c(s)p(s)}{s} \right| \tag{16}$$

Inverting this equation yields

$$\frac{1}{r} = \sup \left| \frac{1}{1 + \dfrac{c(s)p(s)}{s}} \right| = \sup |S(s)| \tag{17}$$

Maximizing the radius of this circle, r, is thus equivalent to minimizing the sensitivity function. In general, it is more important to impose this constraint in some frequency ranges than others; therefore, this objective can be stated in the H^∞ norm as a desire to minimize

$$\| J_2 \|_\infty = \| W_2(s) S(s) \|_\infty \tag{18}$$

where $W_2(s)$ is large in the frequency ranges for which $1/r$, given by (17), is undesirably large.

In real systems, it is often desirable to restrict the bandwidth of the closed-loop system. This is important for two practical reasons. First, it is necessary to neglect the effects of very high-frequency dynamics in a design model to limit the order of the model. As a result, it is undesirable to drive the system in frequency ranges in which the input–output behaviour of the model is not representative of the actual system. The second reason for limiting system bandwidth is to bound the power requirements of the closed-loop system. For minimum-phase systems, the sensitivity minimization theory states that it is possible to achieve 'perfect' disturbance rejection, ($S \cong 0$), and tracking, ($T = 1 - S \cong 1$), over an arbitrarily large frequency range (Helton 1985). This is achieved by allowing the controller bandwidth, and hence the energy requirements, to be as large as necessary.

The bandwidth of the closed-loop system can be restricted by applying a weighting, $W_3(s)$, to the complementary sensitivity function, $T(s)$, that is unbounded as $s \to \infty$ and requiring that

$$\| J_3 \|_\infty = \| W_3(s)\, T(s) \|_\infty \tag{19}$$

be a bounded function. If $W_3(s)$ has a zero excess m at high frequencies, then $T(s)$ and the open-loop transfer function must have at least a pole excess of m for (19) to be finite. This weighting function can be used to regulate the character of the system in the cross-over region and at high frequencies.

4. H^∞-Optimization

The three objective functions discussed above can be combined into a single function:

$$J(c(s), a, b) = \|(a|J_1|)^2 + (b|J_2|)^2 + |J_3|^2\|_\infty \tag{20}$$

where $a, b \in \mathbb{R}^+$ are non-negative scalars which can be adjusted to alter the relative weighting of each term and the optimization is over a class of admissible compensators, $c(s)$. Only two weighting coefficients, a and b, need to be considered since (20) can always be scaled to make the third coefficient equal to 1. It is possible to rewrite (20) and define an optimization problem as

$$\min_{c(s)} J(c(s), a, b) = \min_{c(s)} \|a^2(W_1 S)(W_1 S)^* + b^2(W_2 S)(W_2 S)^* + (W_3 T)(W_3 T)^*\|_\infty \tag{21}$$

where $W(s)^* = W(-s)$, a and b are fixed, and J is optimized over the set of compensators that stabilize the nominal linear system. The optimization of functions of this type over the class of stabilizing compensators has been discussed in Kwakernaak (1985) and Helton (1985). The solution of this optimization problem usually reduces to solving a polynomial equation in which the number of coefficients to be matched equals the number of unknowns. An exception arises when there are multiple solutions to the optimization which yield the same minimum. The solution of these equations yields a set of coefficients for the numerator and denominator polynomials of the compensator. The interested reader should consult Kwakernaak (1985) for the details of the solution procedure for this optimization problem.

5. Example

In this section, the approach outlined previously will be applied to an example problem. In this problem the effects of transport delays will be neglected and it will be assumed that the control variable V can be specified instantaneously. Therefore, it is reasonable to choose the simplest nominal model, namely, $p(s) = 1$. The control objective is to regulate the position of the flame front defined as the point where the temperature profile has the value of 0·5 at the point $x_{\text{desired}} = 0$. It is assumed that there is one sensor that measures the temperature at the desired location of the flame front, $y_1 = T(x_{\text{desired}}, t)$. If $h(y_1) = y_1 = T(x_{\text{desired}}, t)$ the only non-linearity is due to the temperature profile, which from the general shape of the profile shown in Fig. 1, is a saturation-type non-linearity. For simplicity it is assumed that the output of the sensor is scaled such that the derivative $dh(T(x^*))/dx = (dh/dT)(dT/dx)(x^*) = 1$. This assumption is not necessary; it has been included to scale the problem such that the nominal linear model for the plant is an integrator, $1/s$. Thus scaled, the circle criterion can be applied to guarantee stability exactly as discussed in the previous section. Finally, the goal is to design a compensator, $c(s)$, such that the nominal system, $c(s)p(s)/s$, has a closed-loop bandwidth of between 1 and 10 rad/s.

There are three weighting functions that need to be defined. The weighting $W_1(s)$ is chosen to be

$$W_1(s) = \frac{(s+10)^2}{s^2} \tag{22}$$

since a detailed disturbance process has not been postulated. The asymptote of the Bode magnitude plot of this transfer function has a slope of -40 dB/decade at low frequencies and is horizontal with a magnitude of 0·01 above 10 rad/s. By increasing the scalar a in (21) this weighting function can be used to decrease the magnitude of the sensitivity function at each frequency and thus increase the system bandwidth. The second weighting function $W_2(s)$ is given by

$$W_2(s) = \frac{(s+0{\cdot}2)(s+500)}{(s+2)(s+50)} \tag{23}$$

The magnitude asymptotes of this weighting function are 1 at frequencies below and above 0·2 and 500 rad/s, respectively, and 10 between 2 and 50 rad/s. The asymptote has a slope of $+20$ dB/decade between 0·2 and 2·0 rad/s and -20 dB/decade between 50 and 500 rad/s. The rationale for this function is as follows. The magnitude of this weighting function is largest in the frequency range of the desired cross-over frequency, and therefore, in the frequency range where the Nyquist curve of the linear subsystem is in the neighbourhood of the -1 point. In conjunction with the scalar b, this weighting can be used to decrease the magnitude of the sensitivity function in this region. In light of (16) and (17), this amounts to increasing the distance of the nearest approach to the -1 point of the Nyquist locus and thus increasing the tolerance to uncertainty in the measurement h. Finally, the weighting $W_3(s)$ is chosen as

$$W_3(s) = \frac{s+10}{10} \tag{24}$$

This weighting has a zero excess of 1 to ensure that $T(s)$ has a pole of excess of 1. The location of the zero begins attenuation of $T(s)$ near the desired cut-off frequency.

It is now possible to solve the optimization problem (21). Following the method outlined in Kwakernaak (1985), the minimizing compensator is given by

$$c(s) = \frac{10n(s)}{s(s+2)(s+50)d(s)} \tag{25}$$

where $n(s)$ and $d(s)$ are polynomials given by a solution to (3.10) in Kwakernaak (1985):

$$\begin{aligned} &n_1(s)n_1(-s)d(s)d(-s) + n_2(s)n_2(-s)n(s)n(-s) \\ &\qquad = \lambda^2(d_1(s)d(s) + d_2(s)n(s))(d_1(-s)d(-s) + d_2(-s)n(-s)) \end{aligned} \tag{26}$$

where $n_1(s)$, $n_2(s)$ and $d_1(s)$, $d_2(s)$ correspond to the numerators and denominators, respectively, of the weightings on the sensitivity function, S, and its complement, T, and where λ^2 is the optimal value of the objective function. To simplify the notation, a single weighting, $W_\alpha(s)$, is introduced for the sensitivity and a weighting, $W_\beta(s)$, is introduced for its complement:

$$W_\alpha(s) = \frac{n_1(s)}{d_1(s)} = W_1(s) + W_2(s) \tag{27}$$

and

$$W_\beta(s) = \frac{n_2(s)}{d_2(s)} = W_3(s) \tag{28}$$

For the solution of (26), the numerator polynomial, $n(s)$, can be assumed to have degree equal to one less than the number of zeros of $d_1(s)$ plus the number of poles of $p(s)/s$ in the open right-half plane; the degree of the denominator polynomial, $d(s)$, can be assumed to have degree equal to one less than the number of zeros of $d_2(s)$ plus the number of zeros of $p(s)/s$ in the open right-half plane plus the zero excess of $W_\beta(s)$. For the examples presented here, the degree of $n(s)$ and $d(s)$ is 4 and 1, respectively. The numerical algorithm which determines the compensators solves for the coefficients of $n(s)$ and $d(s)$ by matching coefficients in (26). Since there may be several solutions to (26) which are not necessarily optimal, it is necessary to check the solution by showing that it satisfies a sufficient condition (3.16 in Kwakernaak 1985). It turns out to be much easier to solve the necessary condition (26) and check sufficiency.

Figure 5 shows the effect on the complementary sensitivity function, $T = 1 - S$, of increasing the coefficient a of the weighting W_1 while holding $b = 0$. The numbered curves correspond to $a = \{0{\cdot}1, 1{\cdot}0, 10{\cdot}0\}$, respectively. As a is increased, the cross-over frequency of the weighting W_1 is increased. This reduces the magnitude of the sensitivity function at low frequencies and increases the system bandwidth as shown by the fact that the function $T(s)$ rolls off at progressively higher frequencies in Fig. 5.

Figure 6 illustrates the effect of holding $a = 0{\cdot}1$ and increasing b on the Nyquist diagram of the system. The numbered curves correspond to $b = \{0{\cdot}4, 1{\cdot}0, 4{\cdot}0\}$,

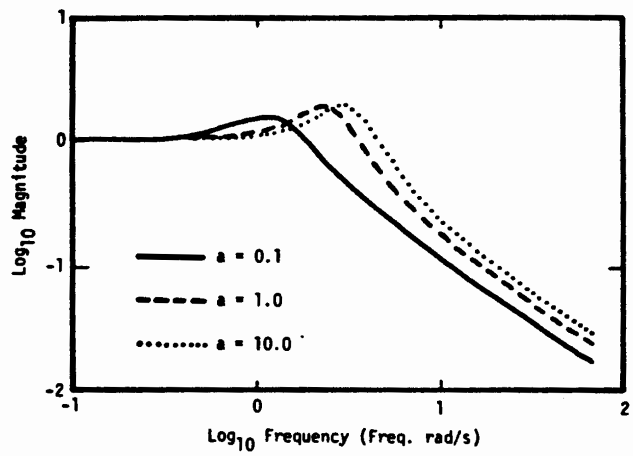

Figure 5. Effect of weighting (a) on the nominal closed-loop sensitivity function.

respectively. Increasing b increases the distance of the nearest approach of the Nyquist contour to the -1 point. Therefore, the radius of the largest circle centred at $(-1, 0)$ and not touching the contour increases. Recalling the discussion of the circle criterion, it is clear that the system can tolerate more uncertainty in the temperature profile. Figure 7 shows the effect on the sensitivity for the same values of b. The important feature of this plot is that the closed-loop bandwidth is reduced as b is increased.

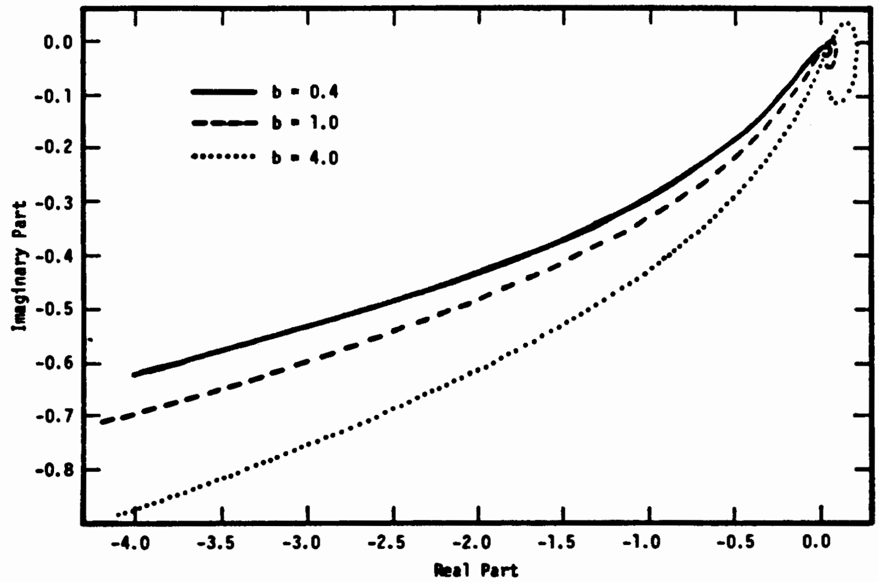

Figure 6. Effect of weighting (b) on the Nyquist diagram of the nominal open-loop transfer function.

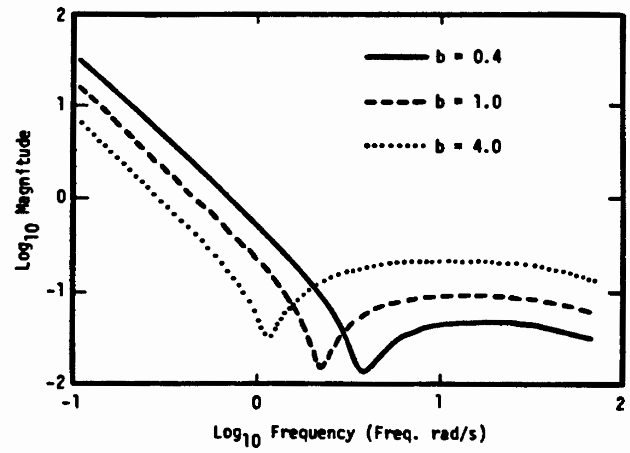

Figure 7. Effect of weighting (b) on the magnitude of the nominal open-loop transfer function.

In this example we have illustrated an approach to the design of compensators for controlling the location of the combustion front in a simple class of reaction-diffusion equations. In the example, an integrator has been chosen to model the process to be controlled. In a more realistic problem, it might be necessary to include the effects of dynamics in the device used to manipulate the fuel–air velocity, V, and possibly to model the time-delay inherent in transport processes. These effects could be included in the nominal model relating the controlled variable to the translation rate of the flame front, $p(s)$. The general approach proposed in the preceding section could still be applied to this more complicated problem.

6. Conclusion

In this paper we have studied the control of a non-linear reaction–diffusion equation representing a flame propagation problem. The control objective was to regulate the location of the flame front in the spatial domain by manipulating the fuel–air velocity. It was shown that the problem could be simplified to that of controlling a linear system with a scalar non-linearity by studying the input–output characteristics of the system. In fact, this point of view enabled us to consider a more general class of systems than modelled by (1) alone. The H^∞-optimality criterion that has been proposed for the compensator design yields a trade-off among performance, power requirements, and stability in the presence of an uncertain non-linearity.

The principle advantage of the approach outlined in this paper is that it is only necessary to understand the qualitative behaviour of the modelled system to control it successfully. This advantage is a necessary feature when developing practical approaches to the control of combustion processes since models are generally inaccurate. Furthermore, the approach does not rely on the use of heuristic or *ad hoc* considerations. Future research must be aimed at studying the applicability of this approach to more complicated flow geometries and combustion models.

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