Understanding the Root Locus Using Gain Plots

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An alternate graphical representation of the Evans root locus plot [1],[2] exposes the relationship between the gain and the pole locations in polar coordinates. The proposed plots, called gain plots, portray the magnitude and angle of each closed-loop system eigenvalue in the complex plane as a function of gain. The visualization is based on the adjustment of a proportional control gain in the same fashion employed in constructing the root locus plot.

Gain plots recast, and in so doing enrich, the information presented in the standard root locus plot, and offer advantages for control system analysis and design. For example, by exposing the correspondence of gain values to specific eigenvalue locations, gain plots are a useful pole-placement tool for achieving closed-loop designs meeting stability and performance specifications. In addition, gain plots reveal by inspection information about the gain sensitivity of closed-loop eigenvalues.

Although the purpose of this article is to present the concept of gain plots and show their use for two example problems, it is interesting to note that they are the result of a natural progression of perspectives of the root locus plot. In fact, the conceptualization of the gain plots from the root locus plot parallels the development (pedagogically) of the Bode plots from the Nyquist diagram. Just as the Bode plots embellish the information of the Nyquist diagram by exposing frequency explicitly in a set of magnitude versus frequency and angle (phase) versus frequency plots, it follows that a pair of gain plots can enhance the standard root locus plot. The development of classical frequency-domain techniques (from the Nyquist diagram to the Bode plots) can be unfolded via a sequence of intriguing three-dimensional representations; in parallel fashion, a sequence of gain-domain methods (from the root locus plot to the gain plots) can be posed [3]. (See [3] for a conceptual framework that motivates the development of the gain plots and shows their utility for single-input, single-output control system analysis and design. A companion paper [4] investigates multivariable systems.)

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Gain Plots

By graphically depicting the behavior of system eigenvalues as an explicit function of gain, gain plots augment the information of the root locus plot. This section summarizes the concept of gain plots by examining a single-input, single-output example problem that provides significant intuition. Subsequently, we investigate a multivariable system for which new behavior is observed.

The root locus plot shows the location of the characteristic roots, i.e., the eigenvalues, in the complex plane as the proportional gain (or any real-valued parameter) is changed. It is assumed that the plant dynamics are given by an input-output transfer function $g(s)$ that is embedded in a unity negative feedback control system with scalar compensator k. The corresponding closed-loop transfer function $g_{CL}(s)$ is given by

$$g_{CL}(s) = \frac{k\,g(s)}{1 + k\,g(s)}. \qquad (1)$$

The system behavior (stability, performance, etc.) of the closed-loop system is determined by the eigenvalues, i.e., the denominator roots of (1). A root locus plot displays the migration of these roots in the complex plane as the gain k varies in the range $0 \le k < \infty$. If the plant transfer function is viewed as the ratio of numerator and denominator polynomials, i.e., $g(s) = n(s)/d(s)$, then the closed-loop transfer function is

$$g_{CL}(s) = \frac{k\,n(s)}{d(s) + k\,n(s)}. \qquad (2)$$

When $k = 0$, the eigenvalues are the open-loop poles, i.e., the roots of $d(s)$. When $k \to \infty$, some of the eigenvalues approach finite transmission zeros, i.e., the roots of $n(s)$, whereas the remaining eigenvalues tend to infinity (in magnitude) in the complex plane in a special pattern known as the Butterworth configuration and at a special rate, discussed later.

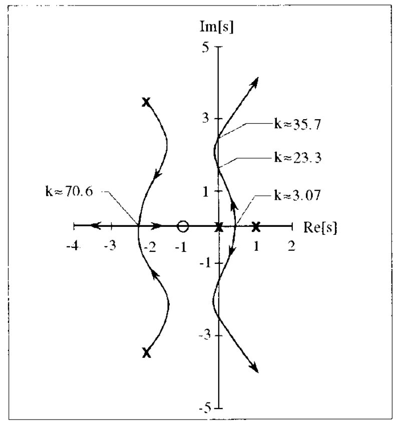

Fig. 1. Root locus plot of (3).

Fig. 1 is the root locus plot for the single-input, single-output example given by the open-loop transfer function

$$g(s) = \frac{(s+1)}{s(s-1)(s^2+4s+16)}. \qquad (3)$$

(Equation (3) is an example problem in [5].) The root locus begins at the open-loop poles located at $s = \{0, +1, -2 \pm 2\sqrt{3}j\}$. The open-loop complex conjugate pole pair migrates to the real axis as the gain is increased. One of these poles then proceeds to the finite transmission zero at $s=-1$; the other pole heads toward an infinite transmission zero along the negative real axis. The two real open-loop poles migrate to $s \approx 0.46$, and then break out from the real axis. As a complex conjugate pole pair, they move to the left of the imaginary axis. Subsequently, they migrate back to the right of the imaginary axis and continue toward infinite transmission zeros along asymptotes of $\pm 60^\circ$. For a small range of gain $23.3 < k < 35.7$, the root locus is located completely within the left half of the complex plane corresponding to a stable closed-loop system. The limits of this range may be found by equating the denominator of (1) to zero and seeking purely imaginary solutions.

An alternative visualization of the root locus plot can be obtained by explicitly graphing the eigenvalue magnitude versus gain in a

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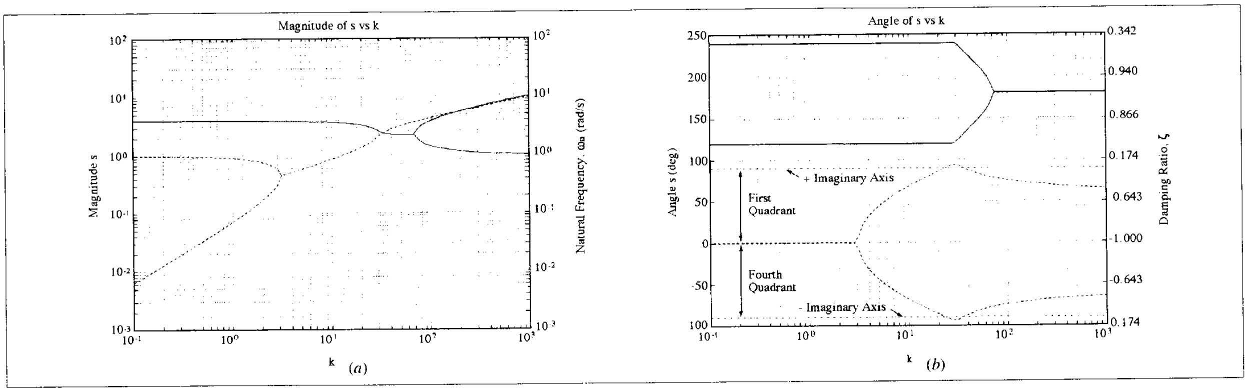

Fig. 2(a) Magnitude versus gain k. (b) Angle versus gain k.

magnitude gain plot and the eigenvalue angle versus gain in an angle gain plot. Fig. 2(a) and (b) shows these plots for the system of (3). The magnitude gain plot employs a log-log scale whereas the angle gain plot uses a semilog scale (reminiscent of the magnitude and phase Bode plots, respectively). In these plots, the solid lines track the loci of the poles that start as a complex conjugate pair; the dotted lines represent the loci of the pole pair that originates on the real axis. (Also shown are the ±90° lines denoting the imaginary axis of the complex plane.)

Information about the open-loop eigenvalues as $k \to 0$ shows i) there is a set of (unstable) open-loop poles at an angle of 0° having magnitudes of 0 and 1, and ii) there is a complex conjugate open-loop pole pair having magnitude 4 at angles of 120° and 240°. For positive values of gain, the system operates under closed-loop negative feedback and the eigenvalue trajectories change in interesting ways. Notice that when a given pole pair is complex, the two poles have the same magnitude but are distinguished in angle. Conversely, when poles lie on the real axis, they have a principal angle of either 180° or 0° corresponding to negative or positive real values, respectively. Thus, a pair of complex conjugate eigenvalues is shown as a single curve in the magnitude gain plot with corresponding angles symmetrically configured about the real axis (0° or 180°). As the gain is adjusted, complex conjugate eigenvalues may become distinct real eigenvalues, causing their angles to become equal (at a multiple of 180°) and permitting their magnitudes to differ.

From the gain plots, break points corresponding to points where branches leave or enter the real axis of the root locus can be observed to occur. For the example, a breakout point occurs at $k \approx 3.07$ and a break-in point occurs at $k \approx 70.6$. For gains greater than the k at the break-out point, the angle gain plot indicates that the loci of the two branch points are not on the real axis and the corresponding single curve of the magnitude gain plot confirms that the trajectories are those of a complex conjugate pair. For gains greater than the k at the break-in point, the angle gain plot indicates that the loci of the two branch points are on the real axis. The corresponding two curves of the magnitude gain plot indicate the distance from the origin to the eigenvalues on the real axis.

The high gain asymptotes of the root locus are found by examining the angle gain plot for large values of k. The finite zero at $s=-1$ is identified by the single pole asymptotically approaching unity magnitude at an angle of 180°. The remaining three eigenvalues asymptotically approach infinite zeros at angles ±60° and 180°. For gains higher than those reported in Fig. 2(a) and (b), these asymptotes are increasingly prominent.

The gain plots highlight several important stability and performance features of the system. Stability may be determined from the angle gain plot by noting if the angles of all eigenvalues map to locations in the second or third quadrants of the complex plane. This corresponds, for example, to the range $90° < \theta < 270°$. For the example, the angle gain plot shows that the system is unstable (with angles indicating first and fourth quadrant eigenvalue locations) for all gains except for the range $23/3 < k < 35.7$. This range of gains may be read directly from the angle gain plot.

In addition to stability, performance measures are presented directly by the gain plots. In particular, the natural frequency, ω_n (rad/s), is the magnitude shown in the magnitude gain plot. If the eigenvalues are on the real axis, the magnitude gain plot presents the system time constants. The damping ratio is given by $\zeta = -\cos\theta$ where θ is the angle shown in the angle gain plot. As illustrated in Fig. 2(a) and (b), supplementary axes can be added to the gain plots displaying ω_n and ζ making this information available directly. (Negative damping ratios correspond to an unstable system.) For instance, the open-loop complex conjugate poles of the example have a natural frequency of 4 rad/s and a damping ratio of 0.5, although this information is "secondary" since the open-loop system is unstable (due to the real poles). Thus, given a design specification for ω_n and ζ, the requisite value of k may be determined by inspection from the gain plots, making them a useful graphical pole-placement tool.

Further inspection of the gain plots provides information about the closed-loop eigenvalue sensitivity to changes in gain. The slopes of the gain plots represent the change in magnitude and angle of each eigenvalue per change in gain. This information is useful in the design of robust control systems that are less sensitive to gain variations. In the example, the system is especially sensitive to gain variations near the break points and when k is small as evidenced by the rapid change in the magnitude of one of the system eigenvalues. Clearly, as $k \to \infty$, i) the angles in the angle gain plot asymptotically approach the Butterworth configuration, and ii) the magnitudes of the eigenvalues are related to the gain via a power law relationship depicted as straight lines on the magnitude gain plot. An in-depth treatment of asymptotic behavior of high gain eigenvalues (and its relation to sensitivity) is covered in [6],[7].

The classical concepts of gain and phase margins, traditional measures of system robustness, are applicable using the gain plots. The gain margin is the factor by which the gain can be increased or decreased for the closed-loop system to reach a stability-instability boundary. This factor can be determined from the angle gain plot by identifying the gain interval for which all eigenvalues have angles within the stable region. For the example, at $k=30$ there is a lower gain margin at $20\log(23.3/30) = -2.20$ dB and an upper gain margin at $20\log(35.7/30) = 1.51$ dB. In principle, a system with a larger gain margin band is relatively more stable than one with a smaller band. The phase margin is the largest angular interval corresponding to unity magnitude gain for which the closed-loop system is closed-loop stable. It can be determined from the angle gain plot by identifying the minimum distance from any of the poles to the unstable region at $k = 1$. These concepts of relative stability are the subject of a separate paper [8].

In addition to the advantages above, including the ability to show the influence of gain directly, gain plots provide a unified approach for single-input, single-output and multivariable systems where compensation dynamics are governed by a single scalar gain amplifying all plant inputs. The advantage of the gain plots to uniquely identify locus branches as a function of gain is of paramount importance in multivariable systems analysis, where this information is typically hidden in multivariable root locus plots.

Multi-Input, Multi-Output Example

This example demonstrates the utility of the gain plots for multivariable systems that possess complex conjugate root loci. The state space representation for this example (from [9] and later used in [10]) is a linearized model of the vertical plane dynamics of an aircraft with three inputs, three outputs and five state variables.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1.132 & 0 & -1 \\ 0 & -0.054 & -0.171 & 0 & 0.071 \\ 0 & 0 & 0.000 & 1 & 0 \\ 0 & 0.049 & 0 & -0.856 & -1.013 \\ 0 & -0.291 & 0 & 1.053 & -0.686 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ -0.120 & 1 & 0 \\ 0 & 0 & 0 \\ 4.419 & 0 & -1.665 \\ 1.575 & 0 & -0.073 \end{bmatrix} u \qquad (4)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x \,. \qquad (5)$$

The dynamics of this fifth order system can be represented by a 3×3 plant transfer function matrix with no finite transmission zeros and open-loop poles at $s = \{0, -0.7801 \pm 1.0296j, -0.0176 \pm 0.1826j\}$. The pole at the origin indicates that the open-loop system is marginally stable. To determine the stability of the closed-loop system, the multivariable root locus shown in Fig. 3 may be drawn under the assumption that the same scalar gain amplifies the three inputs to the transfer function matrix.

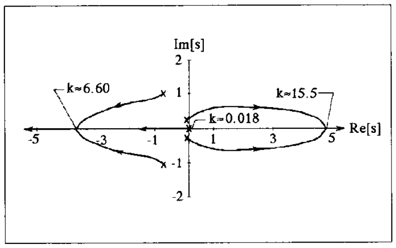

Fig. 3. Multivariable root locus example.

From the root locus, it is not clear if all of the eigenvalues exist in the left-half plane for low values of gain. The root locus suggests that at high values of gain the system is unstable, but fails to indicate the range of gain for which instability occurs. In fact, the closed-loop system is never stable for positive gain values.

Fig. 4(a) and (b) is the pair of gain plots for the multivariable system given by (4) and (5). Several interesting phenomena occur as the gain is increased. For example, the eigenvalue starting at the origin migrates into the right-half plane, indicating instability, reaching a maximum real value of $s \approx 0.010$ at a gain of $k \approx 0.018$. This eigenvalue then returns to the left-half plane becoming marginally stable at a gain of $k \approx 0.043$, at which point the two other eigenvalues whose angles are symmetric about zero are already unstable. This behavior is shown clearly in the angle gain plot of Fig. 4(b), but is obscured in Fig. 3.

The rate at which the eigenvalues migrate towards a magnitude of infinity is seen in the magnitude gain plot. The single eigenvalue that begins at the origin proceeds towards infinity along the negative real axis at a rate proportional to k (the magnitude gain plot shows a high gain slope of unity). This slope is characteristic of a system with one "excess" pole (i.e., possessing one more pole than finite zero). The two complex conjugate pole pairs proceed toward infinity at a rate proportional to $k^{1/2}$ (shown as a high gain slope of 1/2 in the magnitude gain plot), indicative of a system with two "excess" poles [6].

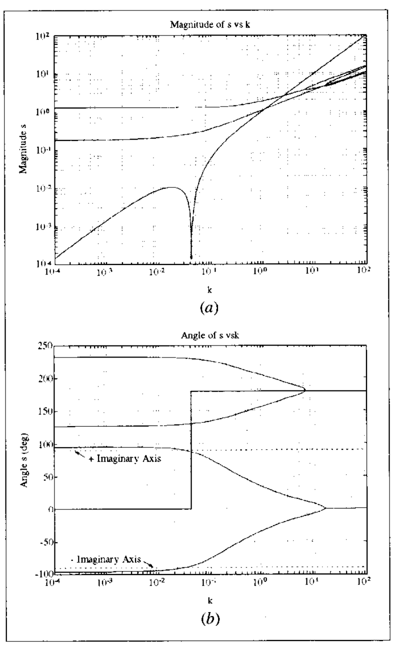

Fig. 4. (a) Magnitude versus gain k. (b) Angle versus gain k.

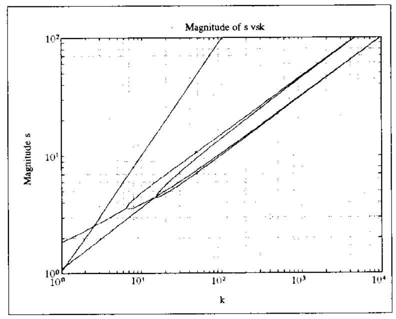

Fig. 5. Expanded magnitude gain plot for multivariable example.

Unusual behavior is exhibited by the complex conjugate pole pairs as they break into the real axis and proceed to $\pm\infty$. Fig. 5 extends the magnitude gain plot of Fig. 4(a) to a gain value of 10^4. The figure confirms that at high gains the magnitude gain plot slopes of the two complex conjugate pole pairs have a value of 1/2. As $k \rightarrow \infty$, there are four parallel lines with the same slopes. This group of four lines may be separated into two sets of identical lines within each set. An interesting phenomenon depicted clearly in Fig. 5 is that the two identical lines are comprised of an eigenvalue magnitude from each of the original complex conjugate pairs. It is as if the complex conjugate eigenvalues have *swapped partners* (or "pole vaulted" each other)!

Closing

The gain plots are a set of illuminating plots that expand and enhance the control engineers' design tool set. By presenting eigenvalue magnitudes and angles in separate graphs, the gain plots simplify and supplement the information contained in the Evans root locus plot, in analogy to the role the Bode plots play with respect to the Nyquist diagram. As such, the gain plots add a gain "dimension" to the Evans root locus plot, just as the Bode plots add an explicit frequency "dimension" to the Nyquist diagram. Strong connections exist among the four graphical controls tools: the Bode, Nyquist, Evans root locus, and the gain plots. The Nyquist diagram and the Evans root locus plot span a two-dimensional complex plane. The Bode plots and gain plots span a three-dimensional (real) space. All the tools are valuable for stability and performance evaluation.

The proposed gain plots enhance the root locus plot by explicitly portraying the relationship between the gain and the location of each eigenvalue whose trajectories are mapped by the root locus. The enhancement enables the control designer to identify, by observation, an eigenvalue location with a specific gain, and hence directly view the influence of the gain on stability as well as on system performance. Furthermore, the gain plots provide a direct measure of eigenvalue gain sensitivity. The change in magnitude and angle of each eigenvalue per change in gain is indicated by the slope of the lines. This measure of sensitivity highlights the "cost" of selecting eigenvalue locations corresponding to specific gain values, and provides the designer with a novel graphical means to assess control system robustness.

We have introduced the concept of gain plots in our undergraduate and graduate controls courses with tremendous success. Students readily capture the basic concept and proof of their excitement is evident in their endorsement of gain plots for a variety of control system analysis and design problems. Common student "feedback" is that "the gain plots give you a feeling of what's going on." As educators, we value the gain plots as offering an alternative and complementary representation of the classical graphical tools. For research, we see the gain plots being valuable for multivariable systems analysis and design, where compensation involving different gains for each input channel can be explored. (As an aside, our original motivation was to understand better the details of multivariable root loci plots. In the process of working out the details, we realized how gain plots fit into the full classical framework leading to their applicability to both single-input, single-output and multi-input, multi-output systems.)

In closing, control engineers have historically embraced powerful graphical design methods with striking success. These methods supply significant insight, permitting rapid system analysis and synthesis. We propose the gain plots as a new graphical design tool whose properties are key to a more complete understanding of classical controls concepts. The gain plots provide a broad spectrum of information about closed-loop control systems, including stability, performance, and robustness attributes, and are a recommended addition to the control engineers' tool set.

References

[1] W.R. Evans, "Graphical analysis of control systems," *Trans. AIEE*, vol. 67, pp. 547-551, 1948.

[2] W.R. Evans, "Control system synthesis by root locus method," *Trans. AIEE*, vol. 69, pp. 14, 1950.

[3] T.R. Kurfess and M.L. Nagurka, "New design paradigms for SISO control system synthesis," Tech. Rep. #EDRC 24-49-91, Engineering Design Research Center, Pittsburgh, PA, 1991.

[4] M.L. Nagurka and T.R. Kurfess, "New design paradigms for MIMO control systems synthesis," Tech. Rep. #EDRC 24-50-91, Engineering Design Research Center, Pittsburgh, PA, 1991.

[5] K. Ogata, *Modern Control Engineering*. Englewood Cliffs, NJ: Prentice-Hall, 1990.

[6] T.R. Kurfess and M.L. Nagurka, "High gain control system design with gain plots," Tech. Rep. #EDRC 24-48-91, Engineering Design Research Center, Pittsburgh, PA, 1991.

[7] T.R. Kurfess and M.L. Nagurka, "A geometric paradigm exposing high gain root sensitivity of single-input single-output systems," Tech. Rep., Engineering Design Research Center, Pittsburgh, PA, 1991.

[8] M.L. Nagurka and T.R. Kurfess, "The range of the gain falls mainly on the s-plane," Dept. of Mechanical Eng., Tech. Rep., Carnegie Mellon Univ., Pittsburgh, PA, 1991.

[9] Y.S. Hung and A.G.J. MacFarlane, "Multivariable feedback: A quasi-classical approach," *Lect. Notes Control and Inform. Sci.*, vol. 40, 1982.

[10] J.M. Maciejowski, *Multivariable Feedback Design*. Reading, MA: Addison-Wesley, 1989.

1992 Conference on Control Applications

The 1992 Conference on Control Applications (CCA) will be held on September 13-16, 1992, at the Stouffer Plaza Hotel in Dayton, OH. The sponsoring society is the IEEE Control Systems Society and cooperating organizations are the local IEEE Dayton Section and the NAECON. The Stouffer Center Plaza Hotel is the premier hotel of Dayton, OH. Located only 12 minutes from Wright Patterson Air Force Base, the hotel is a skywalk away from the Dayton Convention Center.

Technical Program

The technical program will consist of a wide range of topics in theory and applications. The theoretical topics include all areas of decision, control, optimization, and adaptation. For example: specific topics are adaptive control theory, intelligent control, robustness, H-Infinity Methods, Singular Value Decomposition, Variable Structure control, Stochastic and Distributed Systems.

The applied areas include (but are not limited to): Aerospace Applications involving Avionics, Control, Fault Tolerance, Stability, Multivariable Control, Expert Systems, Command, Control, and Communications, Man-Machine and Biomedical Systems, Flexible Systems, Process Control, Modeling and Identification. Other application areas are also strongly encouraged.

Special tutorials are designed to encourage both theoretical and applied researchers to work together to merge theory and practice. For the theoretical people, special tutorialswillbegivenemphasizingtheoretical work. For the applied people, theoretical experts will illustrate aspects of their theoretical research that have a wide range of applicability. In all these tutorials, as with the general conference, the goal will be to merge theory withpractice.

A reception is planned for Sunday night; tours at the world famous Air Force Museum are also scheduled. The deadline for paper submissions is January 15, 1992. For further information contact the General Chairman: Dr. D. W. Repperger, Building 33, AAMRL/BBS, Wright Patterson Air Force Base, Dayton, OH 45433-6573, phone: 513-255-5742, FAX: 513-255-9687, E-Mail: drepperge@AAMRL.AF.MIL.