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Parameter Variation for Dynamic System Performance

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Abstract: The paper promotes the use of special eigenvalue plots for providing a geometric perspective in control analysis and design. It is shown that the proposed eigenvalue plots are related to the classic root sensitivity function, a normalized measure of parameter variation effects on the closed-loop eigenvalues. Following the analytical development, an example demonstrating a position-actuated gantry robot with link compliance is analyzed. The effects of system parameters on the closed-loop performance are investigated using the proposed eigenvalue plots and resulting sensitivity plots.

INTRODUCTION

This paper adopts a global closed-loop dynamic perspective and suggests a set of tools for linear control system analysis and design. The tools provide the designer with insight into the explicit functional dependence of the closed-loop system eigenvalues on various system parameters. The associated plots, called the eigenvalue plots, are an alternate graphical representation of the Evans root locus plot. They explicitly portray the closed-loop eigenvalue magnitude *vs.* a system parameter in an eigenvalue magnitude plot and the eigenvalue angle *vs.* parameter in an eigenvalue angle plot. These two plots permit the designer to visualize the effects of parameter variation on the system performance through eigenvalue relationships.

In classical control theory the root sensitivity, S_p, is defined as the relative change in the system roots or eigenvalues, λ_i (i = 1,..., n), with respect to a system parameter, p. The typical parameter analyzed is the forward proportional controller gain, k, in a closed-loop configuration. The root sensitivity with respect to gain is given by

$$S_k = \frac{d\lambda(k)/\lambda(k)}{dk/k} = \frac{d\lambda(k)}{dk}\frac{k}{\lambda(k)} \tag{1}$$

Since the eigenvalues may occur as complex conjugate pairs, S_k may be complex.

Equation (1) is often introduced in determining the break points of the Evans root locus plot for single-input single-output systems. At the break points, S_k becomes infinite as at least two of the n system eigenvalues undergo a transition from the real domain to the complex domain or vice versa. This transition causes an abrupt change in the relation between the eigenvalue angle $\angle\lambda$ and gain k yielding an infinite eigenvalue derivative with respect to gain (Ogata, 1990).

The root sensitivity function S_k is a measure of the effect of parameter variations on the eigenvalues. It is important since one of the key objectives of feedback control theory is to reduce system sensitivity to variations in parameters. For example, the control system of a robot should be relatively insensitive to the payload carried by the arm for the recommended payload range. If the robot's performance is sensitive to payload variations, then the control system is not robust and performance is difficult to guarantee. In this case, S_m, where m is the payload mass, should be relatively small over the operational range of m. Such considerations are critical if control designers are to develop high performance, robust, closed-loop systems.

This paper presents a geometric technique for finding the root sensitivity which is built upon the set of "gain plots" described above (Kurfess and Nagurka, 1991a). An example is presented demonstrating the insight provided via this geometric perspective on the sensitivity function, and its utility in system design and analysis.

ROOT SENSITIVITY ANALYSIS

This section derives the root sensitivity function by employing a polar representation of the eigenvalues in the complex plane. The derivation is based on the following three assumptions: *(i)* the systems analyzed are lumped parameter, linear time–invariant systems; *(ii)* there are no eigenvalues at the origin of the s-plane, *i.e.*,

$$\lambda_i \neq 0\ ,\ \ \forall i = 1, \ldots, n \tag{2}$$

(although the eigenvalues may be arbitrarily close to the origin singularity); and *(iii)* the forward scalar gain, k, is real and positive, *i.e.*, $k \in \Re$, $k > 0$. From these assumptions, the following observations are made: the real component of the sensitivity function is given by

$$\mathrm{Re}\{S_k\} = \frac{d\ln|\lambda(k)|}{d\ln(k)} \tag{3}$$

and the imaginary component of the sensitivity function is given by

$$\mathrm{Im}\{S_k\} = \frac{d\angle\lambda(k)}{d\ln(k)} \tag{4}$$

where $\angle\lambda$ is the eigenvalue angle.

These observations may be proven by rewriting equation (1) in terms of the derivatives of natural logarithms (Horowitz, 1963; Kuo, 1991) as

$$S_k = \frac{d\ln(\lambda(k))}{d\ln(k)} \tag{5}$$

The natural logarithm of the complex value, λ, is equal to the sum of the logarithm of the magnitude of λ and the angle of λ multiplied by $j = \sqrt{-1}$. Thus, equation (5) becomes

$$S_k = \frac{d\left[\ln|\lambda(k)| + j\angle\lambda(k)\right]}{d\ln(k)} \tag{6}$$

Since j is a constant, equation (6) may be rewritten as

$$S_k = \frac{d\ln|\lambda(k)|}{d\ln(k)} + j\frac{d\angle\lambda(k)}{d\ln(k)} \tag{7}$$

The complex root sensitivity function is now expressed with distinct real and imaginary components employing the polar form of the eigenvalues. It follows from assumption *(ii)* that ln(k) is real. (In general, most parameters studied are real and this proof is sufficient. If, however, the parameter analyzed is complex, it is a straightforward task to extend the above analysis.)

The proof is completed by taking the real and imaginary components of equation (7), yielding equations (3) and (4). It is interesting to note that the Cartesian representation of S_k is related to the polar representation of λ_i.

GEOMETRIC RELATIONS TO GAIN PLOTS

The gain plots (Kurfess and Nagurka, 1991a) explicitly depict the eigenvalue magnitude *vs.* gain in a magnitude gain plot and the eigenvalue angle *vs.* gain in an angle gain plot. The magnitude gain plot employs a log–log scale whereas the angle gain plot uses a semi–log scale (with the logarithms being base 10.) Although gain is used as the variable of interest in the derivation, any parameter may be selected. This capability is pursued in the example which demonstrates the ease with which design parameter explorations can be conducted.

By observation, the slope of the magnitude gain plot is the real component of S_k. The magnitude gain plot slope, M_m, is

$$M_m = \frac{d \log(|\lambda(k)|)}{d \log(k)} \tag{8}$$

which may be rewritten as

$$M_m = \frac{d[\log(e)\ln(|\lambda(k)|)]}{d[\log(e)\ln(k)]} = \frac{d \ln(|\lambda(k)|)}{d \ln(k)} \tag{9}$$

corresponding to equation (3).

Furthermore, the slope of the angle gain plot is linearly related to the imaginary component of S_k by the constant, $(\log(e))^{-1}$. The angle gain plot slope, M_a, is

$$M_a = \frac{d \angle\lambda(k)}{d \log(k)} \tag{10}$$

which may be rewritten as

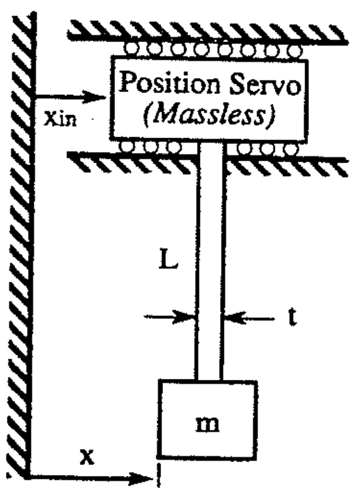
$$M_a = \frac{d \angle\lambda(k)}{d[\log(e)\ln(k)]} = \frac{1}{\log(e)} \frac{d \angle\lambda(k)}{d \ln(k)} \tag{11}$$

Hence, M_a is proportionally related to equation (4) with constant $(\log(e))^{-1}$ (Kurfess and Nagurka, 1991b).

EXAMPLE

This section highlights the utility of the eigenvalue plots and the associated sensitivity analyses for system design by exposing the influence of a parameter on the closed-loop dynamics. The example demonstrates the determination of the root sensitivity function with respect to a system parameter. In so doing, the method extends the "gain plots" and the root locus to system variables other than the control gain, enabling the control designer to conduct parametric sensitivity studies when synthesizing and analyzing dynamic systems.

This example considers the translational dynamics of an industrial gantry-robot carrying a payload m via a single cantilevered link of length L, base width w, and thickness (height) t. The plant and nominal system parameters are shown in Figure 1.



Nominal Values

E = 70 GPa
m = 5 kg
w = 5 cm (into paper)
L = 1 m
$\zeta = 0.02$

Fig. 1: Lumped Parameter Model of Gantry Robot.

The transfer function relating the input gantry displacement, x_{in}, to the payload displacement, x, is

$$\frac{X(s)}{X_{in}(s)} = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{12}$$

where the natural frequency, ω_n, and damping ratio, ζ, are

$$\omega_n = \sqrt{\frac{K}{m}} \quad , \quad \zeta = \frac{b}{2\sqrt{Km}} \tag{13a,b}$$

in terms of payload mass, m, lumped damping, b, and lumped stiffness, K. From linear elastic beam theory, the stiffness can be expressed as

$$K = \frac{3EI}{L^3} \tag{14}$$

where E is the Young's modulus of elasticity, L is the length of the beam, and I is the area moment of inertia. (For the link of rectangular cross section, $I = wt^3/12$.) The model assumes that all modes above the first mode are well beyond the bandwidth of the controller and are, thus, negligible. The plant is embedded in a unity gain feedback configuration with (forward) proportional control k yielding a closed-loop transfer function

$$g_{cl}(s) = \frac{X(s)}{X_d(s)} = \frac{k\left(2\zeta\omega_n s + \omega_n^2\right)}{s^2 + (1+k)2\zeta\omega_n s + (1+k)\omega_n^2} \tag{15}$$

where X_d is the desired or reference command payload displacement. In typical control design, the proportional gain k is varied to investigate its influence on the closed-loop eigenvalues of a fixed system.

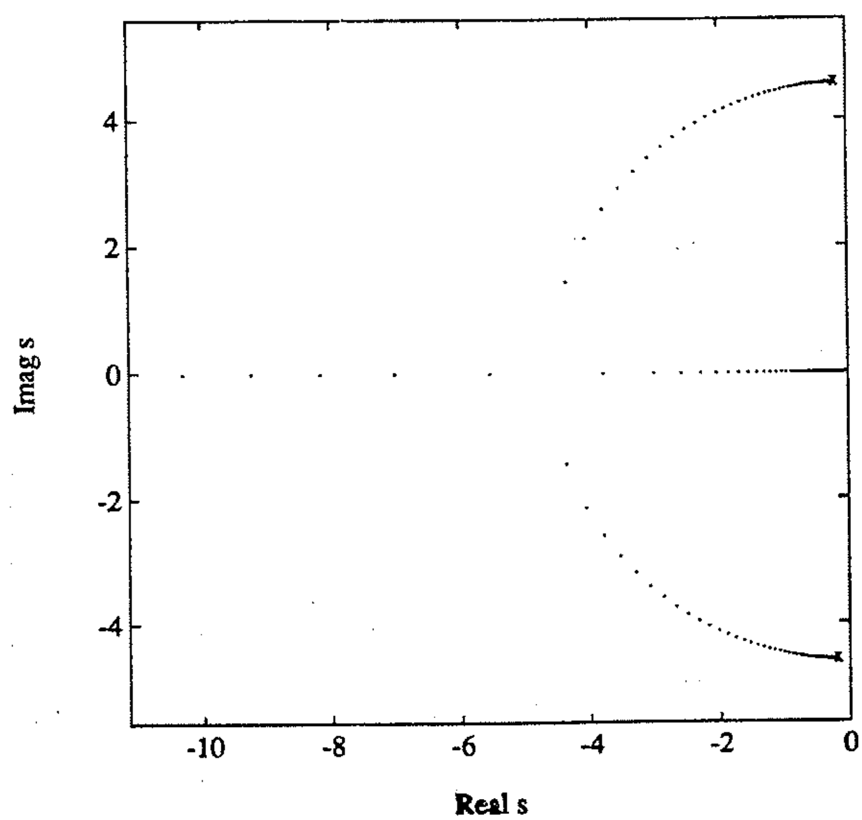


Fig. 2: Root Locus Plot for Example.

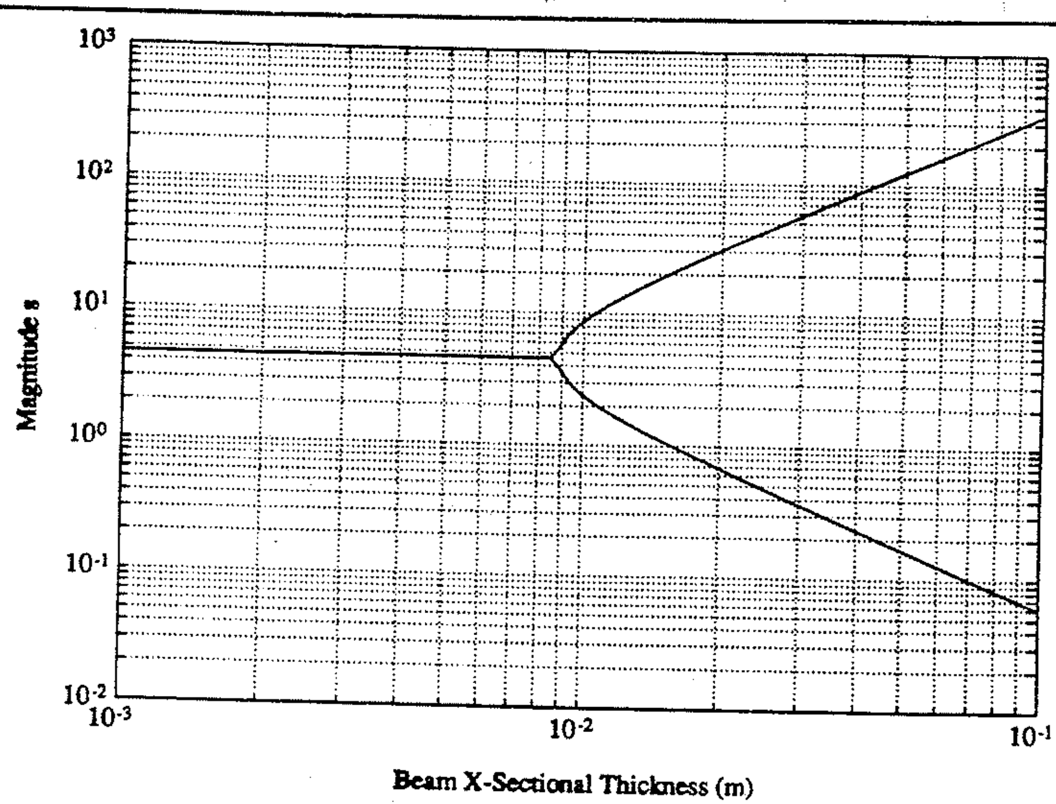


Fig. 3a: Eigenvalue Magnitude Plot for Example.

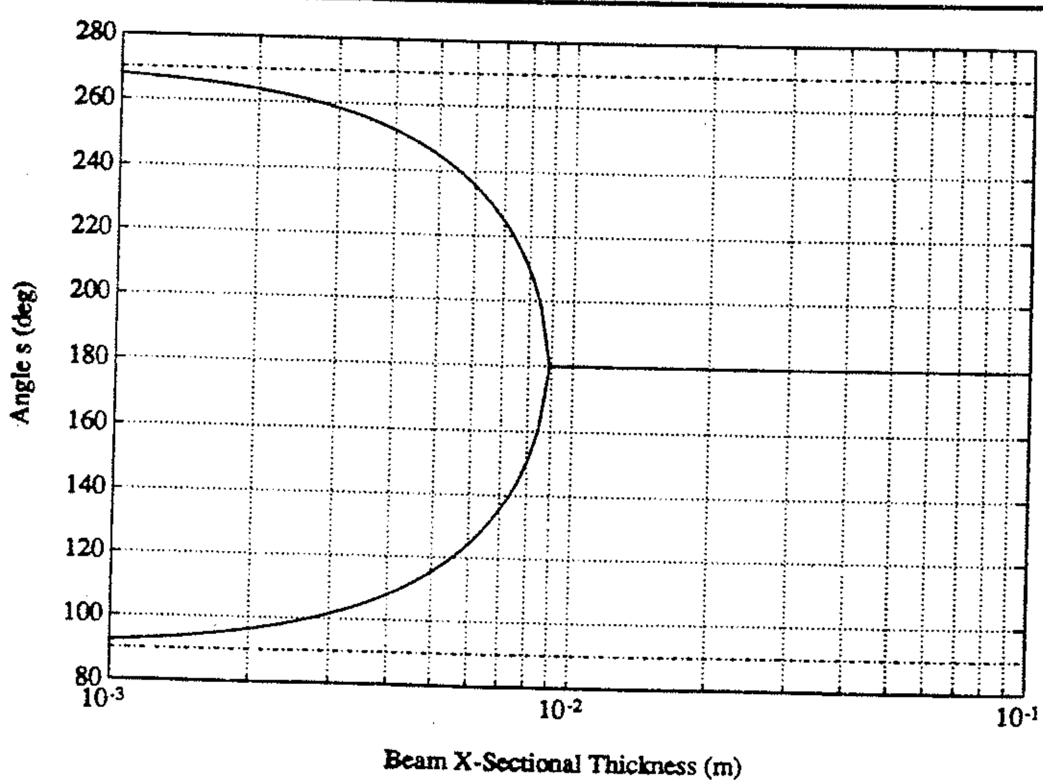


Fig. 3b: Eigenvalue Angle Plot for Example.

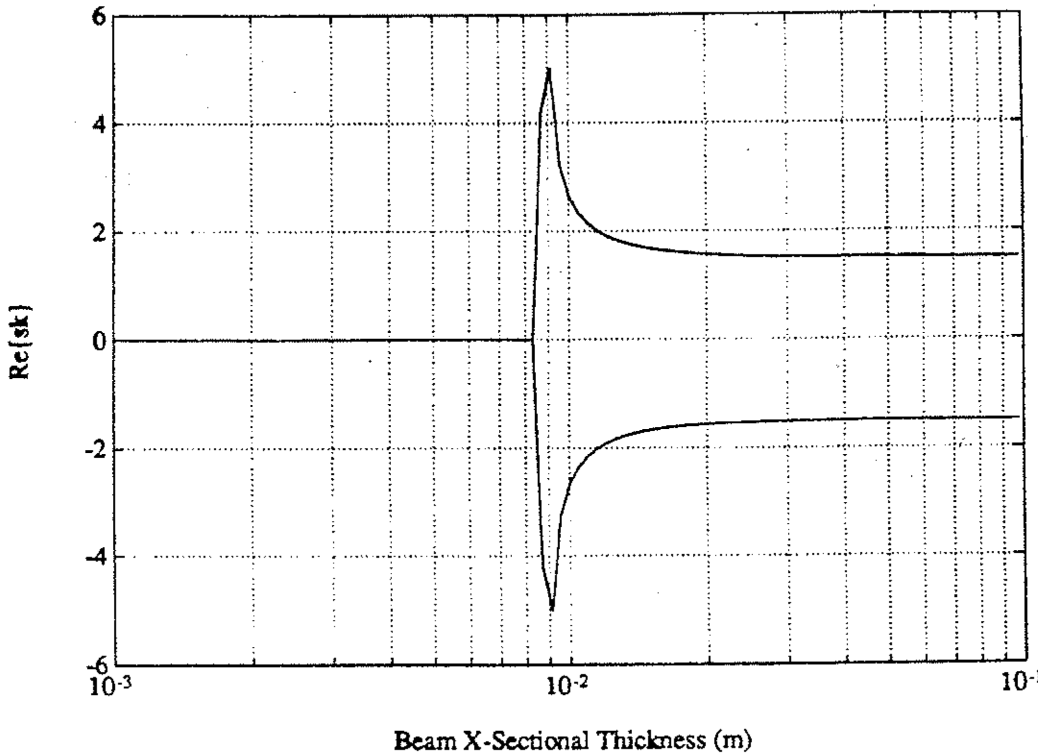


Fig. 4a: Real Component of Sensitivity Function for Example.

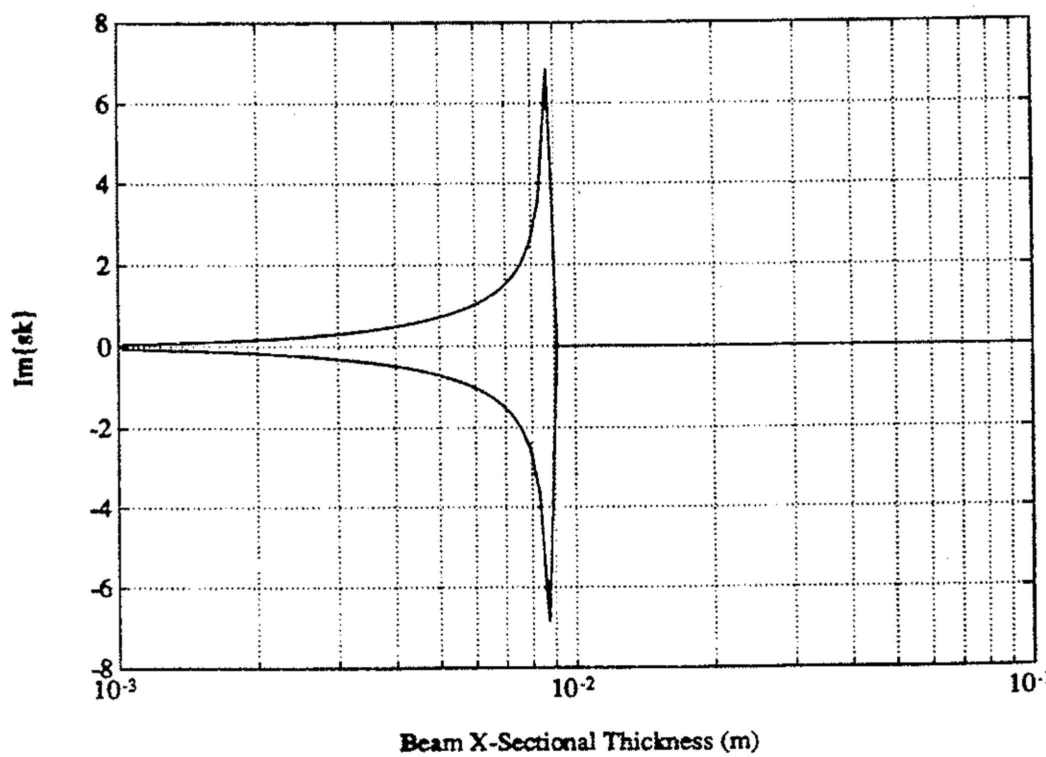


Fig. 4b: Imaginary Component of Sensitivity Function for Example.

The intent of this example is to analyze the influence of geometry on the closed-loop system. In particular, the forward gain is fixed at $k = 20$, and all dimensions except for the thickness are held at their nominal values (as indicated in Figure 1). The resulting root locus plot where t is the implicit variable is shown in Figure 2. At small values of thickness, the locations of the complex conjugate eigenvalues indicate that the closed-loop system is highly oscillatory. At larger thicknesses, the system becomes less oscillatory, and the eigenvalues migrate toward the real axis. At a certain thickness, the eigenvalues break-in to the real axis. As the thickness increases further, the eigenvalues migrate along the real axis; one moves toward negative infinity, while the other approaches the finite transmission zero at the origin.

An alternate perspective is offered in the eigenvalue magnitude *vs.* thickness plot of Figure 3a, and the eigenvalue angle *vs.* thickness plot of Figure 3b. Figures 3a,b show that the thickness corresponding to the break-in point is t_{crit}=8.6 mm. Thicknesses below this value correspond to oscillatory closed-loop behavior; thicknesses above this value correspond to exponentially decaying behavior. The horizontal line of Figure 3a, indicating constant magnitude, demonstrates that thicknesses less than t_{crit} give rise to complex conjugate eigenvalues that follow a circular trajectory about the origin. This is observed in Figure 2. At large thicknesses, the eigenvalue magnitudes behave in an asymptotic manner.

Figures 4a,b are the real and imaginary components, respectively, of the sensitivity function. These plots were obtained from the slopes of Figures 3a,b. The most prevalent feature of these plots is the high sensitivity at t_{crit}. (Theoretically, the sensitivity is infinite at t_{crit} from equation (7). In the plots, finite values are indicated due to the discrete nature of the computation.) At the break-point the high sensitivity is indirectly available by noting the large spacing of the eigenvalues in Figure 2 (where each dot represents a logarithmically equally-spaced thickness increment). The sensitivity plots demonstrate this information in a direct fashion and directly indicate the value of t_{crit}.

The zero real sensitivity below t_{crit} in Figure 4a indicates that the natural frequency (which is the magnitude of the eigenvalues) is not changing as a function of thickness. Well above t_{crit} the high speed time constant (purely real eigenvalue) has a large constant real sensitivity of 1.5. Similarly, the slower eigenvalue has a constant real sensitivity of −1.5 at large thicknesses. This negative sensitivity is a result of $\lambda_i \rightarrow 0$ as $t \rightarrow \infty$. Beams with thicknesses larger than 1 cm possess eigenvalues with highly sensitive real components. Figure 4b shows a reverse trend for the imaginary component of the sensitivity function. Above t_{crit} the imaginary sensitivity is zero, indicating that damping is not a function of thickness. At values of thickness well below t_{crit}, the imaginary sensitivity is also zero. For thicknesses above 1 mm and below t_{crit}, the imaginary sensitivity increases substantially and indicates high damping variations with respect to thickness.

CLOSING

The concept of root sensitivity in classical controls is often introduced to emphasize the high "sensitivity" of eigenvalues with respect to a system parameter such as gain near the break–points. Normally, the root sensitivity function is not discussed as a complex quantity in control system analysis and design. Here, we have derived and demonstrated a powerful means to visualize the root sensitivity function via the eigenvalue plots. The slopes of the eigenvalue plots provide a direct measure of the real and imaginary components of the root sensitivity, and are available by inspection. The use of the eigenvalue plots in conjunction with other traditional graphical techniques offers the designer important information for selection of appropriate system parameters.

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