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EXPLICIT PARAMETER DEPENDENCY IN DIGITAL CONTROL SYSTEMS

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ABSTRACT

This paper introduces a graphically-based tool for the analysis and design of linear, dynamic, digital control systems that is an alternate representation of the digital root locus. The technique exposes the explicit influence of system parameters, such as the forward proportional control gain, on system response characteristics, including natural frequency and damping ratio. In particular, digital gain plots are proposed to elucidate the connection between the control gain and stability/response parameters. These plots provide useful design insight, as demonstrated in two examples.

INTRODUCTION

For continuous, linear, time-invariant, dynamic systems, the root locus method (Evans, 1948) has rightfully earned its place as a mainstay tool for control system analysis and design. An alternate representation of the root locus, called gain plots (Kurfess and Nagurka, 1991), has been proposed for the selection of control gain (or any system parameter) in continuous systems. The premise of the gain plot concept is that eigenvalue-based information, such as natural frequency and damping ratio, is more naturally presented as an explicit dependent function of a system parameter. The availability of the explicit connection between system response characteristics and system parameters enables designers to make selections that meet specifications and uncover design tradeoffs.

The root locus concept is extendible to digital control systems (Franklin, et al, 1990; Houpis and Lamont, 1992; Kuo, 1980; Palm, 1983). Although the mechanics of drawing the root locus are exactly the same in the z-plane as in the s-plane, the pole locations have different meanings, and interpretation for system stability and dynamic response for digital systems is difficult. This paper describes an alternate representation of the digital root locus that lends itself to an easier understanding of stability and dynamic response. The proposed digital gain plots are similar to the continuous time gain plots, and are useful as a design tool in that they highlight the direct mapping between a parameter, such as control gain, and the digital system response.

As noted, while root locus plots for continuous and digital systems are plotted in the same manner, the interpretation of closed-loop pole locations is different. In order to clarify these differences, this paper first reviews the relationship between *(i)* response parameters and *(ii)* continuous and digital root loci.

MAPPING OF ROOT LOCUS FROM CONTINUOUS TO DISCRETE TIME

The Z-transform variable, z, can be expressed in terms of the Laplace variable, s, as $z=\exp(-sT)$ where T is the digital sampling time. By substituting the definition of the Z-transform in terms of the Laplace transform into expressions for root locations in the s-plane, root locations associated with particular response characteristics can be mapped from the s-plane to the z-plane. The mapping of system response parameters from the s to the z planes is shown graphically in Figure 1. Key quantities associated with stability and dynamic response are described below. The important results are derived in the Appendix.

Time constant

In the s-plane, the real part of the root is the negative reciprocal of the response envelope time constant. In the z-plane, the magnitude of the root is the exponential of the sampling time multiplied by the negative reciprocal of the time constant.

Damping ratio

In the s-plane, the damping ratio is equal to the cosine of the angle of the root taken from the negative real axis. In the z-plane. the relation is more complicated. By substituting $z=\exp(-sT)$ into the expression for damping ratio in the s-plane,

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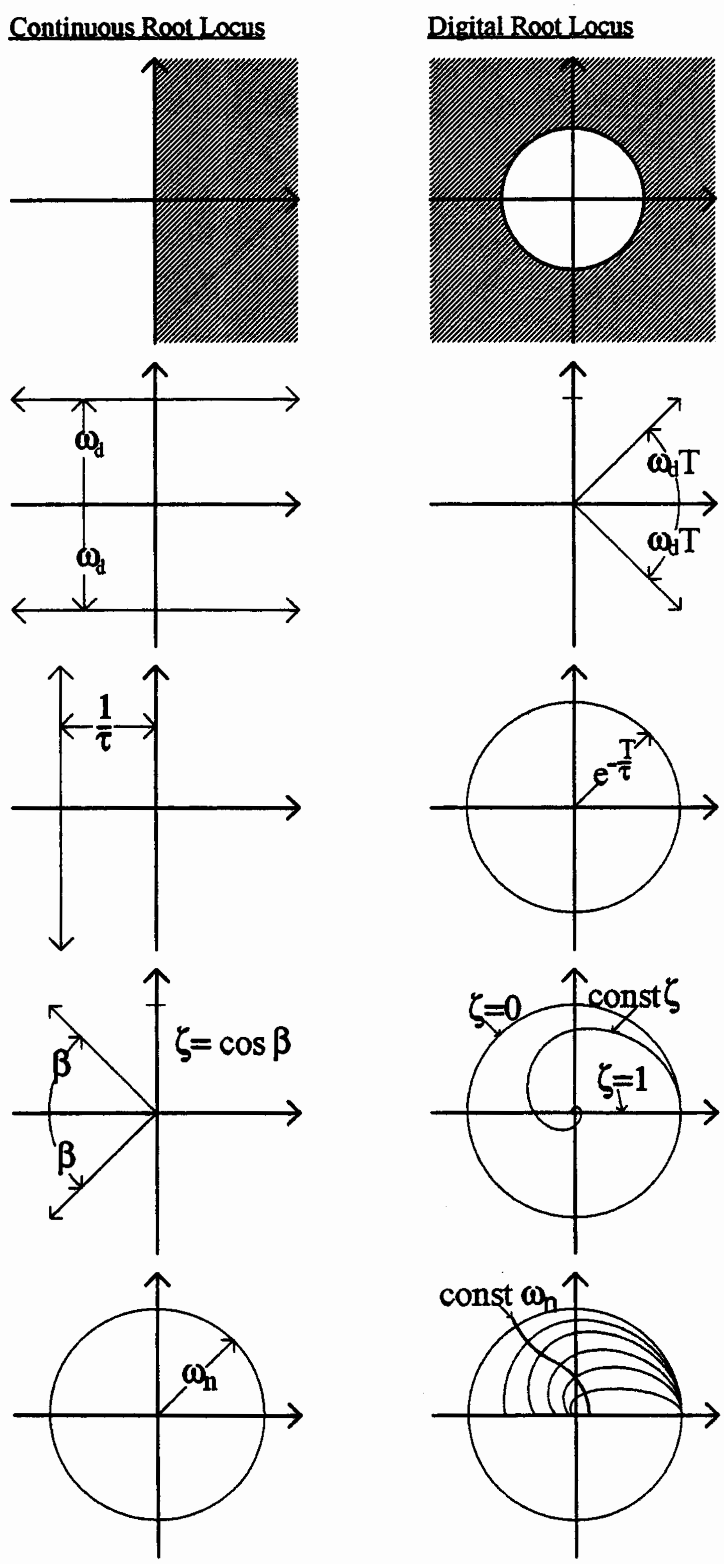

Region of Stability

s-plane: $Re(s) < 0$

- Left half plane

z-plane: $|z| < 1$

- Unit circle

Constant Damped Natural Frequency

s-plane: $Im(s) = \omega_d$

- Horizontal lines

z-plane: $\angle z = \omega_d T$

- Radial lines

Constant Time Constant

s-plane: $Re(s) = -\frac{1}{\tau}$

- Vertical lines

z-plane: $|z| = e^{-T/\tau}$

- Concentric circles

Constant Damping Ratio

s-plane: $\cos(\beta) = \zeta$

- Radial lines

z-plane: $|z| = e^{-\zeta \angle z/\sqrt{1-\zeta^2}}$

- Logarithmic spirals

Constant Natural Frequency

s-plane: $|s| = \omega_n$

- Concentric circles

z-plane:

$\angle z = \sqrt{\omega_n^2 T^2 - \ln(|z|)^2}$

- Orthogonal to const. ζ

Figure 1. Graphical representation of the mapping from the s-plane to the z-plane

an equation for the damping ratio in the z-plane can be obtained. The resulting relation equates the magnitude of the root to a function of the angle of the root and the damping ratio as follows:

$$|z| = \exp\left(-\zeta \angle z / \sqrt{1-\zeta^2}\right) \quad (1)$$

When the angle of the root is increased while the damping ratio is held constant, the magnitude decreases exponentially with the angle. By considering different damping ratios, a set of constant damping ratio lines can be found in the form of concentric logarithmic spirals. The positive real axis corresponds to a unity damping ratio.

Natural frequency

In the s-plane, the undamped natural frequency is equal to the magnitude of the root. The corresponding relation in the z-plane, not commonly presented in textbooks, can be obtained by equating the angle of the root from the positive real axis to a function of the magnitude of the root, the sampling time, and the undamped natural frequency, as follows:

$$\angle z = \sqrt{\omega_n^2 T^2 - (\ln(|z|))^2} \quad (2)$$

When plotted, the lines of constant natural frequency are orthogonal to the lines of constant damping ratio.

Several of these quantities are difficult to determine by inspection from the digital root locus since the relations between root location and the response parameters are complicated. For example, although for a particular root the stability, time constant, and damped natural frequency are readily available from the digital root locus, it is difficult to determine the damping ratio and undamped natural frequency without drawing an additional set of graduated lines. In addition, the lines of constant natural frequency and damping ratio are very close together near the origin and the circumference of the unit circle, and are difficult to discern. Many textbooks (Franklin, et al, 1990; Kuo, 1980; Palm, 1983) contain design examples in which a gain must be selected to obtain a desired response. The response requirements are mapped onto the root locus, and the appropriate areas are masked off, leaving an available design "segment." The magnitude criterion is then used to determine the range of gains for a viable design. This procedure gives limited intuitive feel for the design and often camouflages design tradeoffs.

GAIN PLOTS

As noted, gain plots have been proposed as an alternative representation of the root locus plot for continuous time systems. Gain plots depict eigenvalue locations in terms of polar coordinates (magnitude and angle) as explicit functions of a parameter, such as forward loop gain. In doing so, the plots directly map the gains corresponding to particular system response characteristics. In particular, since the magnitude of a complex conjugate eigenvalue in the complex plane is directly related to the natural frequency and the angle is directly related to the damping ratio, reporting these quantities enables the direct identification of the gains required to obtain particular performance specifications. Other information is available from these plots, including root sensitivity (Kurfess and Nagurka, 1992), robustness measures (Nagurka and Kurfess, 1992), and design tradeoffs.

It is possible to generate similar plots for discrete time systems. By plotting the magnitude of the roots of the closed-loop transfer function in the z-plane *vs.* gain, the stability and, indirectly, the time constant are available. The angle of the roots gives the damped natural frequency.

Since many design problems are stated in terms of the damping ratio and undamped natural frequency, an alternate set of plots is considered that show these quantities as a function of gain. Since they are orthogonal on the z-plane, the damping ratio and natural frequency are independent parameters that can be used to completely describe root locations. It is also useful to plot the reciprocal of the product of the damping ratio and natural frequency. This quantity gives the response envelope time constant, and is equivalent to plotting the sampling time divided by the negative natural log of the magnitude of the root location.

MAPPING EQUATIONS

The natural frequency and damping ratio for a particular root can be determined algebraically. They can be expressed in terms of the angle and magnitude of the root location, z, and the sampling time, T, as follows:

$$\zeta = -\frac{\ln(|z|)}{\sqrt{(\angle z)^2 + (\ln(|z|))^2}} \qquad \omega_n = \frac{\sqrt{(\angle z)^2 + (\ln(|z|))^2}}{T} \quad (3)$$

These relations are derived in the Appendix.

In following the format of continuous time gain plots, the independent variable, gain, is plotted on a logarithmic scale. The damping ratio is plotted on a linear scale, for $1 \le \zeta \le 1$, and the natural frequency and time constant are shown on logarithmic scales. Mathematically, the time constant can be in the range $-\infty \le \tau \le +\infty$; negative values correspond to unstable systems.

EXAMPLES

Example 1: First Order System

Consider a first order plant with the continuous open loop transfer function:

$$G(s) = \frac{1}{s+1} \quad (4)$$

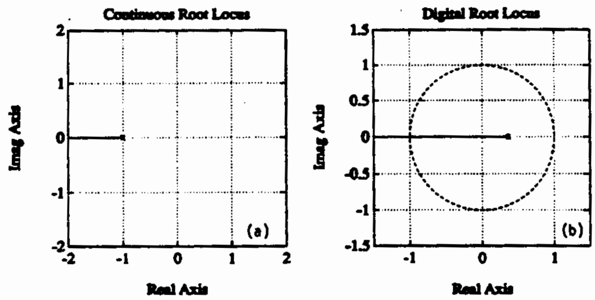

Figure 2. Continuous and digital root loci for example 1

With sampling time T=1, the corresponding digital open loop transfer function is:

$$G(z) = \frac{0.632}{z - 0.368} \tag{5}$$

The root locus plots for the continuous and digital systems embedded in proportional control loops are shown in Figure 2. The plots indicate that the continuous system is stable for all gains whereas the digital system is stable only for low gains, eventually becoming unstable as the root locus leaves the unit circle. In fact, all digital systems with more poles than zeros become unstable at high gains since branches of the excess poles migrate toward infinity. Another feature indicated in Figure 2 is that the digital root locus crosses from the positive to negative real axis with the response becoming oscillatory as it does. The digital root locus plot does not indicate the gains that make the system unstable nor that make the response oscillatory.

The gain plots for the digital system are shown in Figures 3a,b. The natural frequency as an explicit function of gain is plotted in Figure 3a. It shows a low gain asymptote of 1 rad/s, a high gain asymptote tending to infinity, and a "resonant peak" of infinite frequency at a gain of 0.578. The damping ratio plot as a function of gain is shown in Figure 3b. It shows that the system becomes unstable (negative damping ratio) at a gain of 2.16 corresponding to the stability boundary. It also shows that the damping ratio is unity for gains below 0.578 indicating that the system is oscillatory only for gains above this value. Figure 3c is a plot of the time constant *vs.* gain. The time constant is the reciprocal of the product of natural frequency and damping ratio, and, as such, this figure is determined from Figures 3a and 3b. Figure 3c shows that the response time of the system dips to zero at a gain of 0.578. This corresponds to the deadbeat gain, where the system, when subjected to a step input, reaches steady state after one time step. The deadbeat gain is easily identified from the time constant plot.

Figures 4a-d show a series of simulated digital system unit step response plots for different gains. Figure 4a is the response for a gain of 0.2. As expected, the response is non-oscillatory. From the time constant plot, the system has a time constant of 0.70 at this low gain. Since the system reaches 98% of the steady state value after about 4 time constants, the settling time of this system is approximately 2.8 seconds.

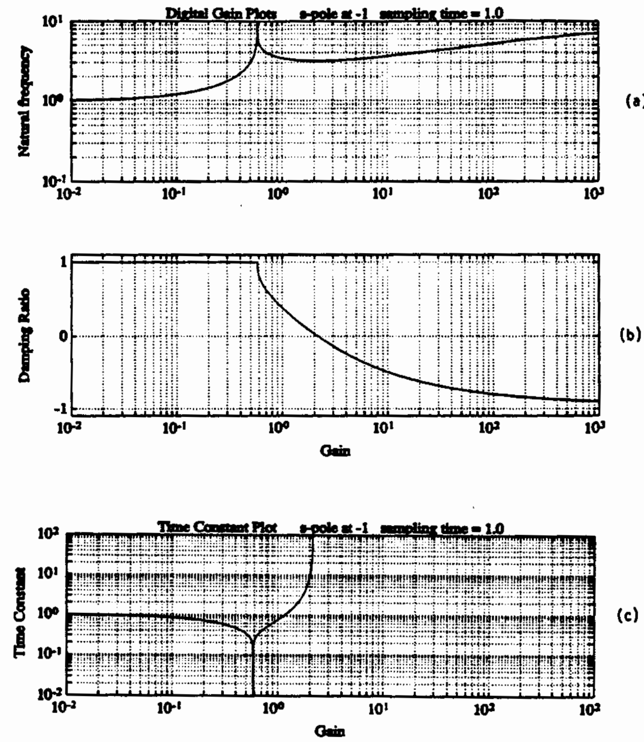

Figure 3. Digital gain plots for example 1

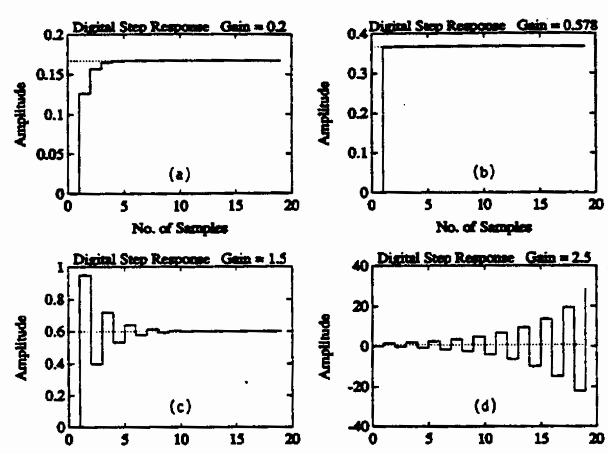

Figure 4. Unit step responses for digital system in example 1

Figure 4b is the response for the deadbeat gain, and shows that the system reaches steady state after one time step corresponding to the fastest possible system. Figure 4c is for a higher, but still stable, gain of 1.5. As predicted from the gain plots, the system response becomes oscillatory. The response of Figure 4d for a gain of 2.5 confirms that the system becomes unstable at gains above 2.16.

This example illustrates some of the basic properties of the digital gain plots. In order to more fully demonstrate the value of the digital gain plots, another, more complicated example is presented.

Example 2: Third Order System

This example considers a third order unstable plant with the continuous open loop transfer function:

$$G(s) = \frac{s+0.5}{(s^2+2s+2)(s-0.5)} \tag{6}$$

With a sampling time of 0.2, the corresponding digital open loop transfer function is:

$$G(z) = \frac{(z+0.9352)(z-0.9048)}{(z^2-1.6048z+0.6703)(z-1.1052)} \tag{7}$$

The root locus plots for the continuous and digital systems with proportional control are shown in Figure 5. The continuous time plot shows that the system is open-loop unstable, but becomes stable above a critical (yet unidentified) gain. For the digital system, the question of stability is not easily answered. Again, the open-loop system is unstable, since one root starts outside the unit circle. As the gain is increased, this root crosses into the unit circle. However, the complex conjugate pair of roots that starts inside the unit circle leaves the unit circle as the gain is increased. The root locus plot does not identify the gain at which the single root enters the unit circle nor the gain at which the pair leaves the unit circle. If the pair leaves before the single root enters, the system will never be stable, but if the single root enters first, the system will be stable for a range of gains. In summary, from the digital system root locus, it is not clear what, if any, gains make the system stable.

The corresponding gain plots of Figures 6a,b show the natural frequency and damping ratio of each root as a function of gain. The single real root is represented by the solid line, while the complex conjugate pair is represented by the dashed and dotted lines. Figure 6c shows the time constant as a function of gain. These plots give physical insight into the system that is otherwise unavailable from the root locus. For example, the damping ratio plot shows that the system is stable for gains between 2.0 and 11.5. Within this range of gains, the plots show that the natural frequency of all roots increases and the damping ratio of the complex pair decreases as the gain is increased. The plots also show that there is a design tradeoff between damping ratio and settling time in the range of gains between 2.0 and 5.88, and that there is a minimum settling time at a gain of 5.88. In summary, the plots are useful for selecting gains meeting specific design requirements. This feature is examined in the system response simulations of Figure 7.

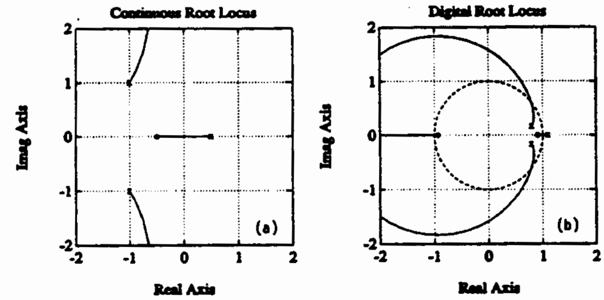

Figure 5. Continuous and digital root loci for example 2

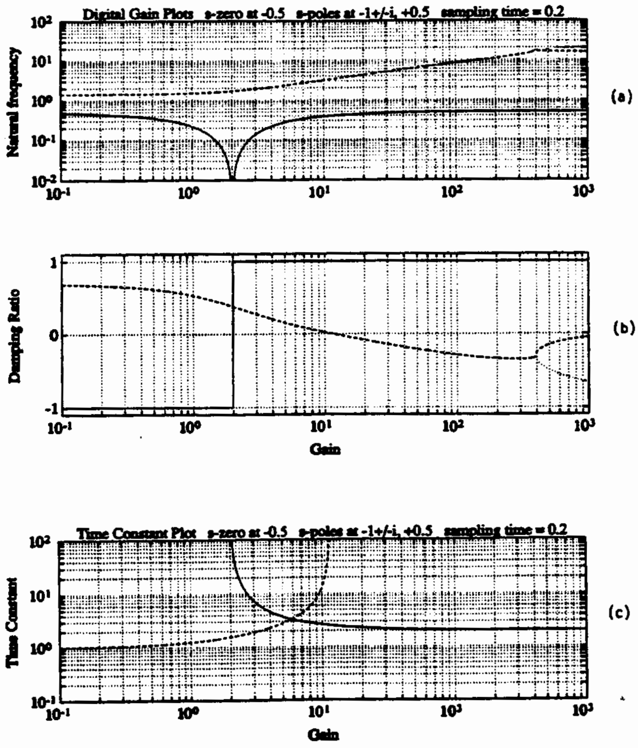

Figure 6. Digital gain plots for example 2

Figures 7a-f show a series of simulations of the unit step response for various gains. Figure 7a shows an unstable response for a gain of 1.5. Figure 7b shows a stable response for a gain of 2.5 just above the lower stability boundary. At this gain the response is mainly non-oscillatory. The single real root is dominant, as confirmed from the time constant plot, where, at a gain of 2.5, the real root (represented by the solid line) has a much longer time constant than the complex conjugate pair.

Figure 7c represents the response that is as quick as possible, with the oscillatory part of the response decaying in less than 8 seconds. To meet this requirement, the time constant of the complex conjugate pair must be at most 8/4=2 seconds. From the time constant plot, this corresponds to a gain of 3.29.

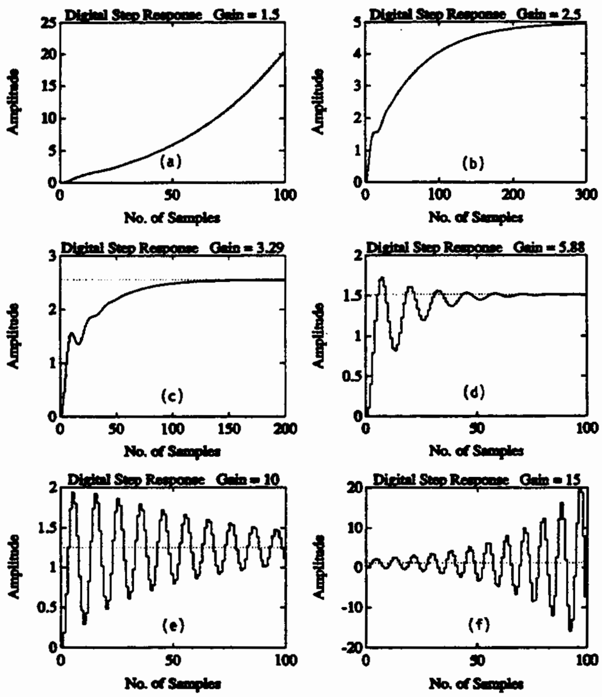

Figure 7. Unit step responses for digital system in example 2

The lines on the time constant plot cross at a gain of 5.88. At this gain, the system has the shortest settling time. The response is shown in Figure 7d for this fastest system. The single real part and the complex conjugate part are equally dominant, and die out after the same time. This response is quicker than the other responses.

Figure 7e is for a higher gains of 10 still in the region of stability. The settling time becomes larger, and the oscillatory root becomes more dominant. When the gain is increased to 15 outside the range of stability, the system becomes oscillatory and unstable, as shown in the response of Figure 7f.

CONCLUSIONS

Digital gain plots are proposed as an alternate representation of the digital root locus. The plots unfold the information of the digital root locus in much the same way that the continuous gain plots recast the information of the continuous root locus, *i.e.*, useful quantities are calculated from the root locations and plotted as an explicit function of gain. In the case of the digital gain plots, these useful quantities are natural frequency, damping ratio, and time constant.

The work reported here has not taken into account variations in the sampling time. It would be useful to develop a design tool to help determine both the gain and the sampling time required to meet a set of design requirements for a digital system. Including the sampling time as a parameter adds another dimension to the problem. Three-dimensional gain plots can be generated with the sampling time and gain as the independent variables. These plots contain a large amount of information and provide a good intuitive feel for the system, but may be difficult to read. Alternatively, multiple 2 dimensional plots can be generated either as a set of layers of a 3 dimensional plot, or with each giving specific information such as the location of the stability boundary for a range of gains and sampling times.

The digital gain plots used in conjunction with the digital root locus enrich the information presented to the controls design engineer in the sense that the gains corresponding to the stability boundaries, natural frequency and damping ratio can be determined by inspection. From the time constant plot, it is clear which roots are dominant. For simple systems, the deadbeat gain can be determined, and for higher order systems, the gain at which the response will be fastest can be identified by inspection. Most importantly, since these parameters are available directly, the digital gain plots serve as a useful design tool in selecting gains to meet response requirements and in uncovering design tradeoffs.

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APPENDIX - DERIVATIONS

Based on the relation z=exp(−sT), roots in the s-plane, s=a+bi, can be mapped to the z-plane.

Stability Region

In the s-plane: $s = a+bi$
stable if $a<0$

In the z-plane: $z = \exp((a+bi)T)$
$z = \exp(aT)(\cos bT + i\sin bT)$

So exp(aT) is the magnitude of the z root. Then if a<0,

$$|z| = \exp(aT) < 1 \tag{A-1}$$

Lines of constant damped natural frequency, ω_d

In the s-plane: $s = a+bi$
$b = \omega_d$

In the z-plane: $z=\exp(aT)(\cos bT+i\sin bT)$

So bT is the angle of the z root, measured from the positive real axis. Then, if $b=\omega_d$,

$$\angle z = \omega_d T \tag{A-2}$$

Lines of constant response envelope time constant, τ

In the s-plane: $s = a+bi$
$a = -1/\tau$

In the z-plane: $z=\exp(aT)(\cos bT+i\sin bT)$

So exp(aT) is the magnitude of the z root Then, if $a = -1/\tau$,

$$|z| = \exp(T/\tau) \tag{A-3}$$

Lines of constant damping ratio, ζ

In the s-plane: $s = a+bi$
$\zeta = \cos(\angle s) = \cos(\tan^{-1}(b/a))$

$$\zeta = -a/\sqrt{a^2+b^2} \tag{A-4}$$

Lines of constant natural frequency, ω_n

In the s-plane: $s = a+bi$
$\omega_n = |s|$

$$\omega_n = \sqrt{a^2+b^2} \tag{A-5}$$

Multiplying equations (A-4) and (A-5) together,

$$a = -\zeta\omega_n$$

Substituting into equation (A-4), and solving for b,

$$b = \omega_n\sqrt{1-\zeta^2}$$

Substituting these expressions for a and b into s = a+bi,

$$s = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2}$$

In the z-plane:

$$z = \exp(-\zeta\omega_n T)(\cos(\omega_n T\sqrt{1-\zeta^2}) + i\sin(\omega_n T\sqrt{1-\zeta^2}))$$

In terms of the angle and magnitude of z,

$$|z| = \exp(-\zeta\omega_n T)$$
$$\angle z = \omega_n T\sqrt{1-\zeta^2} \tag{A-6}$$

Now, to obtain lines of constant damping ratio, first rewrite equations (A-6) in terms of ω_n.

$$\omega_n = \ln(|z|)/(-\zeta T)$$
$$\omega_n = \angle z/(T\sqrt{1-\zeta^2}) \tag{A-7}$$

Equating, and solving for |z|,

$$|z| = \exp(-\zeta\cdot\angle z/\sqrt{1-\zeta^2}) \tag{A-8}$$

This represents a logarithmic spiral as ζ is held constant and angle is varied.

Mapping Equations

Now solve equation (A-8) for ζ.

$$\ln(|z|)\sqrt{1-\zeta^2} = -\zeta\angle z$$

$$\boxed{\zeta = -\frac{\ln(|z|)}{\sqrt{(\angle z)^2 + \ln(|z|)}}} \tag{A-9}$$

This is the first mapping equation.

To obtain lines of constant natural frequency, plug equation (A-9) into the first of equations (A-7)

$$\omega_n = \frac{\sqrt{(\angle z)^2 + \ln(|z|)^2}}{\ln(|z|)T}\ln(|z|)$$

$$\boxed{\omega_n = \frac{\sqrt{(\angle z)^2 + \ln(|z|)^2}}{T}} \tag{A-10}$$

This is the second mapping equation.

Or, solve for the angle to obtain an expression in terms of the magnitude and the natural frequency

$$\angle z = \sqrt{(\omega_n T)^2 - \ln(|z|)} \tag{A-11}$$

This represents a line orthogonal to the lines of constant ζ as magnitude is varied and ω_n is held constant.