

Robust PI Controller Design Satisfying Gain and Phase Margin Constraints

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Abstract

This paper presents a control design algorithm for determining PI-type controllers satisfying specifications on gain margin, phase margin, and an upper bound on the (complementary) sensitivity for a finite set of plants. Important properties of the algorithm are: (i) it can be applied to plants of any order including plants with delay, unstable plants, and plants given by measured data, (ii) it is efficient and fast, and as such can be used in near real-time to determine controller parameters (for on-line modification of the plant model including its uncertainty and/or the specifications), (iii) it can be used to identify the optimal controller for a practical definition of optimality, and (iv) it enables graphical portrayal of design tradeoffs in a single plot (highlighting tradeoffs among the gain margin, complementary sensitivity bound, low frequency sensitivity and high frequency sensor noise amplification).

1 Introduction

Many methods for tuning PI and PID controllers have been developed and extensive research in design techniques continues driven by the strong use of such controllers in industry. In summarizing the various tuning methods reported, two categories emerge depending on the types of specifications that the design must satisfy. One class of methods emphasizes gain and phase margin specifications, whereas a second class focuses on sensitivity specifications.

Design techniques based on satisfying gain and phase margin specifications include the following. Ho et. al. [1], [2] developed simple analytical formulae to tune PI and PID controllers for commonly used first-order and second-order plus dead-time plant models to meet gain

and phase margin specifications. Ho et. al. [3], [4] reported tuning formulae for the design of PID controllers that satisfy both robustness and performance requirements. Crowe and Johnson [5] presented an automatic PI control design algorithm to satisfy gain and phase margin based on a converging algorithm. Suchomoski [6] developed a tuning method for PI and PID controllers that can shape the nominal stability, transient performance, and control signal to meet gain and phase margins.

Gain and phase margin specifications may fail to guarantee a reasonable bound on the sensitivity, a very important control design property. This point has been considered by several researchers. Ogawa [7] used the QFT-framework to propose a PI design technique that satisfies a bound on the sensitivity for an uncertain plant. Poulin and Pomerleau [8] developed a PI design methodology for integrating processes that bounds the maximum peak resonance of the closed-loop. The peak resonance constraint is equivalent to bounding the complementary sensitivity, which can be converted to bounding the sensitivity. Cavicchi [9] gave a design method for bounding the sensitivity while achieving desired steady-state performance. The method can also be applied to measured data. However, plant uncertainty is not considered, and the procedure fits a simple compensation structure. Crowe and Johnson [10] reported a design approach to find a PI/PID controller that bounds the sensitivity while satisfying a phase margin condition. Kristiansson and Lennartson [11] emphasized the need to bound the sensitivity and complementary sensitivity. They suggested the use of an optimization routine to design PI and PID controllers with low-pass filters on the derivative gain to optimize for control efforts, disturbance rejection and bound on the sensitivity. They also provided tuning rules for non-oscillatory stable plants and plants with a single integrator. Astrom et. al. [12] described a numerical

method for designing PI controllers based on optimization of load disturbance rejection with constraints on sensitivity and weighting of set point response.

Other investigators have pursued research into tuning methods. Yeung et. al. [13] presented a non-trial-and-error graphical design technique for controller design of lead-lag structure that enables simultaneous fulfillment of gain margin, phase margin and crossover frequency.

These papers and many others apply gain and phase margin constraints in finding PI and PID controller designs. Some add limitations on the (complementary) sensitivity. However, there are several differences between approaches reported in the literature and the idea proposed here. First, the approach here bounds the sensitivity of the closed-loop transfer function for all frequencies, not just at the crossover frequencies where the gain and phase margins are satisfied. (It is possible that the gain and phase margin conditions are met with a given PI design, but the sensitivity can be very high.) Second, the approach developed here accounts for plant uncertainty, in that the controller design must satisfy the specifications for a set of plants. Third, the approach presented provides explicit equations to determine the set of all possible controllers. Fourth, with this method it is possible to extract the optimal control design solution for many practical optimization criteria. Fifth, the algorithm can be applied to many types of plants, including continuous and discrete plants, plants with pure delay, non-minimum phase plants, and stable and unstable plants. Sixth, since the algorithm uses explicit equations, and not optimization routines, it is very fast.

2 Problem Statement

Consider an open-loop transfer function, $L(s)$,

$$L(s) = C(s)P(s), \quad (1)$$

where $P(s)$ is a member of a finite set of plants, $P_1(s), \dots, P_n(s)$, and $C(s)$ is a PI controller,

$$C(s) = \frac{a(1 + bs)}{s}, \quad (2)$$

with integral gain a and proportional gain ab .

The gain margin (GM) and phase margin (PM) conditions, the typical measures of robustness, are replaced by a condition on the closed-loop sensitivity inequality,

$$\left| \frac{1}{1 + kL(s)} \right| \leq M \quad \text{for } s = j\omega, \forall \omega \geq 0, k \in [1, K], \quad (3)$$

where the sensitivity bound $M > 1$ and the gain uncertainty of the plant, k , is in the interval $[1, K]$. It can be shown [14] that when $\arg L(j\omega) = -\pi$ rad, then (3)

requires $|L(j\omega)| \leq (M - 1)/M$ for $K = 1$, and thus the GM for a given K is at least

$$GM = 20 \log_{10}(K) + 20 \log_{10} \left(\frac{M}{M - 1} \right). \quad (4)$$

Similarly, when $|L(j\omega)| = 1$, (3) requires $\arg L(j\omega) > -\pi + 2 \arcsin [(2M)^{-1}]$, and thus the PM is at least

$$PM = 2 \arcsin \left(\frac{1}{2M} \right). \quad (5)$$

Inequality (3) is a more encompassing measure of robustness than GM and PM . It places a bound on the sensitivity at all frequencies, not just at the two frequencies associated with the two margins. As an example of the problem of using the GM and PM to capture robustness, consider the following open-loop transfer function,

$$L(s) = k \frac{1.39s + 211}{s} \frac{100}{s(1 + s/1000)} e^{-0.0004s}, \quad k \in [1, 10].$$

whose PM and GM are, respectively, 35 deg and 24 dB for $k = 1$. These values are the minimum margins corresponding to $M = 1.66$ and $K = 10$ in (3). At these M and K values the phase margin is guaranteed to be at least 35 deg for any gain, k , in the interval $[1, 10]$. However, from a frequency-domain analysis of $L(s)$, the PM for $k = 10$ is calculated to be 13 deg and the GM is 4 dB. Thus, the margin specifications fail to guarantee the satisfaction of the PM for all plant gain uncertainties.

The design problem of interest is to find all (a, b) pairs that satisfy (3) for all $P(s) \in [P_1(s), \dots, P_n(s)]$. For plants that include at least one integrator, the sensitivity is proportional to $1/a$ at low frequencies, and for any plant the sensor noise at the plant input is amplified by ab at high frequencies. As such, it is of particular interest to find the pair (a, b) for which a is maximum, i.e., the controller design corresponding to lowest sensitivity at low frequencies, and its associated lowest b .

3 Main Results

To determine the (a, b) values for which the closed-loop system is stable and (3) is satisfied, consider first the special case of no gain uncertainty, i.e., $K = 1$, and a single plant $P(s)$. Splitting $P(s)$ for $s = j\omega$ into its real and imaginary parts,

$$P(j\omega) = A(\omega) + jB(\omega), \quad (6)$$

and substituting it and (2) into (3) gives,

$$\begin{aligned} Da^2 (1 + b^2\omega^2) + 2aA - 2ab\omega B \\ + 1 - M^{-2} \geq 0, \quad \forall \omega \geq 0, \end{aligned} \quad (7)$$

where $D = A^2 + B^2$. For an (a, b) pair which is on the boundary region of the allowed (a, b) values, there exists ω such that (7) is an equality. Moreover, since at that particular ω , (7) is minimum, its derivative (with respect to ω) at the same ω is zero. Thus,

$$\begin{aligned} [2E(1 + b^2\omega^2) + 2D\omega b^2]a + 2\dot{A} \\ - 2b(\omega\dot{B} + B) = 0 \end{aligned} \quad (8)$$

where $E = A\dot{A} + B\dot{B}$ and the dot indicates derivative with respect to ω . Solving (8) for a gives

$$a = \frac{-\dot{A} + b\omega\dot{B} + bB}{E + Eb^2\omega^2 + D\omega b^2} \quad (9)$$

Substituting (9) into the equality of (7) gives a fourth-order equation for b as a function of ω .

$$x_4b^4 + x_3b^3 + x_2b^2 + x_1b + x_0 = 0 \quad (10)$$

where¹, with $Q = 1 - M^2$,

$$\begin{aligned} x_4 &= (QE^2 - 2B\dot{B}E + D\dot{B}^2)\omega^4 + \\ &\quad (-2B^2E + 2QED)\omega^3 + (-DB^2 + QD^2)\omega^2 \\ x_3 &= (2B\dot{A}E + 2A\dot{B}E - 2D\dot{A}\dot{B})\omega^3 + \\ &\quad (2ABE + 2A\dot{B}D)\omega^2 + 2ABD\omega \\ x_2 &= (D\dot{B}^2 + D\dot{A}^2 + 2QE^2 - 2A\dot{A}E - 2B\dot{B}E)\omega^2 \\ &\quad + (2QED - 2A\dot{A}D - 2B^2E + 2D\dot{B}B)\omega + DB^2 \\ x_1 &= (2B\dot{A}E + 2A\dot{B}E - \\ &\quad 2D\dot{A}\dot{B})\omega - 2D\dot{A}B + 2ABE \\ x_0 &= -2A\dot{A}E + D\dot{A}^2 + QE^2 \end{aligned}$$

The allowed (a, b) region for a given M value can be calculated as follows: For a given ω solve (10) for b . Noting that b has four solutions (for a given ω), select the positive real solution for which the resulting closed-loop system is stable and (3) is satisfied for $K = 1$. Then use (9) to find its corresponding a . Searching over a range of frequencies ω enables the boundary of the (a, b) region to be identified. Note that for point (a, b) on the boundary, one of the following conditions can occur: (i) increasing a is inside the region, (ii) decreasing a is inside the region, or (iii) neither increasing nor decreasing a is inside the region. Thus, for two points, (a_1, b) and (a_2, b) , on the boundary, any $a \in [a_1, a_2]$ and b is a pair within the region only if (i) increasing a_1 is within the region, (ii) decreasing a_2 is within the region, and (iii) there exist no (a, b) points on the boundary for any $a \in (a_1, a_2)$.

¹The equations were derived using Matlab's Symbolic Mathematics Toolbox from MathWorks, Inc., Natick, MA.

Remark 3.1 A PI controller exists if and only if there exists a frequency for which an (a, b) pair solving (10) can be found for which the resulting closed-loop system is stable and (3) for $K = 1$ is satisfied. If a PI controller does not exist and is required, try increasing M .

3.1 Example 1: Simplified DC Motor

Consider the plant,

$$P(s) = \frac{1}{s(1 + s/10)} \quad (11)$$

which can represent a simplified model of an armature-controlled DC motor with the input being motor current and the output being speed. For this plant

$$\begin{aligned} A(\omega) &= \frac{-10}{(100 + \omega^2)}, \quad \dot{A}(\omega) = \frac{20\omega}{(100 + \omega^2)^2} \\ B(\omega) &= \frac{-100}{\omega(100 + \omega^2)}, \quad \dot{B}(\omega) = \frac{100(100 + 3\omega^2)}{\omega^2(100 + \omega^2)^2} \end{aligned}$$

Fig. 1 depicts the (a, b) values for the particular case of $M = 1.46$, which is equivalent to a 40 deg PM or greater and a 10 dB GM or greater. (The (a, b) values fall in both shaded regions.) Fig. 1 can also be used to find the (a, b) values which satisfy any GM constraint. For example, if 6 dB gain margin uncertainty is desired (that is $K = 2$), then for any b , the allowed a values should be 6 dB less in order to cope with the increase in uncertainty. The (a, b) region will therefore be the lower shaded region depicted in Fig. 1 where the upper curve is shifted down by 6 dB. The maximum a for $K = 1$ occurs at $(a, b) = (18.2 \text{ dB}, 0.67)$ and maximum a for $K = 2$ occurs at $(a, b) = (12.2 \text{ dB}, 0.67)$, giving the controller designs corresponding to lowest sensitivity at low frequencies. Note that if gain uncertainty K is required then a solution is guaranteed only if there exists at least a single b corresponding to a range of a values in an interval $[a_1, a_2]$ such that $a_2/a_1 \geq K$.

The solution for several M values for plant (11) is depicted in Fig. 2. Each curve is the boundary of the allowed (a, b) values for a given M . The corresponding PM and GM values indicated are the minimum values along the boundary curve, i.e. the PM and GM are equal or greater along the curve.

3.2 Example 2: Loaded DC motor

Let the plant be a loaded DC motor whose transfer function from current to speed includes an integrator, a real pole, a complex zero and pole, and a delay, where the complex zero is smaller than the complex pole,

$$P(s) = \frac{e^{-sT}}{s(1 + s/p)} \frac{1 + 2d_1\omega_1/s + \omega_1^2/s^2}{1 + 2d_2\omega_2/s + \omega_2^2/s^2}$$

with values $p = 200 \text{ rad/sec}$, $d_1 = 0.07$, $\omega_1 = 100 \text{ rad/sec}$, $d_2 = 0.1$, $\omega_2 = 150 \text{ rad/sec}$ and $T = 0.001 \text{ sec}$.

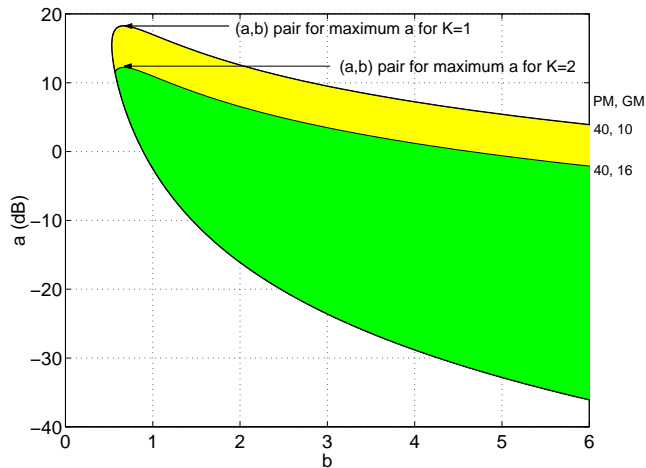


Figure 1: Region of (a, b) values for $M = 1.46$, equivalent to 40 deg phase margin (PM) or greater and 10 dB gain margin or greater (GM for $K = 1$) for Example 1. Lower shaded region is for $M = 1.46$ with additional 6 dB plant gain uncertainty ($K = 2$) for a total of 16 dB or greater.

The (a, b) region for $M = 1.46$ is shown in Fig. 3. The (a, b) pair for maximum a is $(a, b) = (61 \text{ dB}, 0.034)$. For these values, the Nichols plot of the open-loop transfer function is shown in Fig. 4. The fact that the curve touches the shaded region confirms satisfaction of inequality (3).

3.3 Extension to Complementary Sensitivity

Replacing the sensitivity margin constraint (3) by the complementary sensitivity,

$$\left| \frac{kL(j\omega)}{1 + kL(j\omega)} \right| \leq M, \forall \omega \geq 0, k \in [1, K] \quad (12)$$

it can be shown that $L = L_0$ satisfies (3) if and only if $L = \frac{M^2}{M^2-1}L_0$ satisfies (12). This leads to the following Corollary: If (a, b) is a pair that solves the problem stated in Section 2, then the pair

$$\left(\frac{M^2 - 1}{M^2} a, b \right)$$

solves the same problem where (3) is replaced by (12).

4 Optimization

The answer to the question “Which is the best (a, b) pair?” of course depends on the optimization criterion. Seron and Goodwin [15] note that “In general, the process noise spectrum is typically concentrated at low frequencies, while the measurement noise spectrum is typically more significant at high frequencies.” It follows that an optimal controller can be found by weighting the performance at low frequencies and of noise at

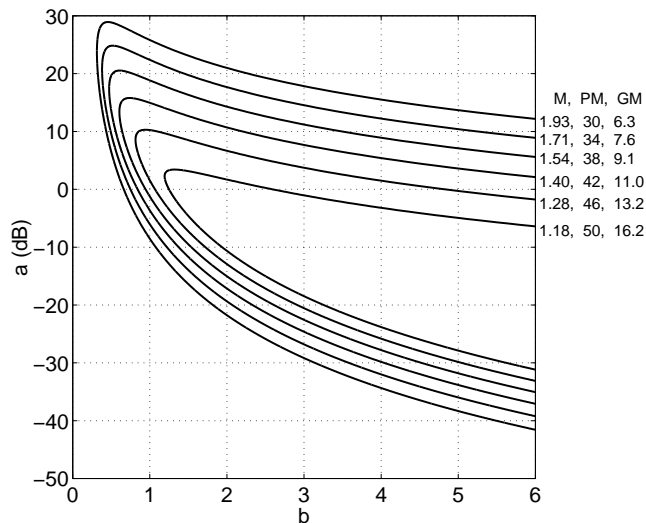


Figure 2: Boundary curves of (a, b) region that satisfy $|\frac{1}{1+L}| < M$ for Example 1. Marked on the right of each curve is its M value, minimal phase margin (PM) and minimal gain margin (GM in dB for $K = 1$) according to (4), and (5).

high frequencies. Since the high frequency noise is proportional to ab and low frequency performance to a , the optimal solution must lie on the boundary of the (a, b) curve. Moreover, if there exists more than one boundary pair for the same a , the one with the lowest ab will be the best. The same condition appears in Kristiansson and Lennartson [11] who proposed several evaluation criteria, one being the ability of the system to handle low frequency load disturbance, represented here by parameter a .

4.1 Example 1: Continued

Consider again the plant of Example 1 with the requirement that there be at least 40 deg PM ($M = 1.46$) and at least 6 dB plant gain uncertainty ($K = 2$). From a plot of a versus ab , shown in Fig. 5, and obtained using the same algorithm to generate the (a, b) plot of Fig. 1, the following conclusions can be drawn.

1. If high frequency sensor noise amplification must be less than 7 dB, then there exists no solution.
2. If high frequency sensor noise amplification greater than 14.7 dB is allowed, then the lowest sensitivity at low frequencies is achieved for $a = 18.3$ dB.
3. For a required low frequency sensitivity, choose the (a, ab) pair for which ab is minimal for the smallest high frequency sensor noise effect.

5 Extension to Uncertain Plants

Assume that the plant, $P(s)$, is known to be one of a finite set of plants, $P_1(s), \dots, P_n(s)$. The controller de-

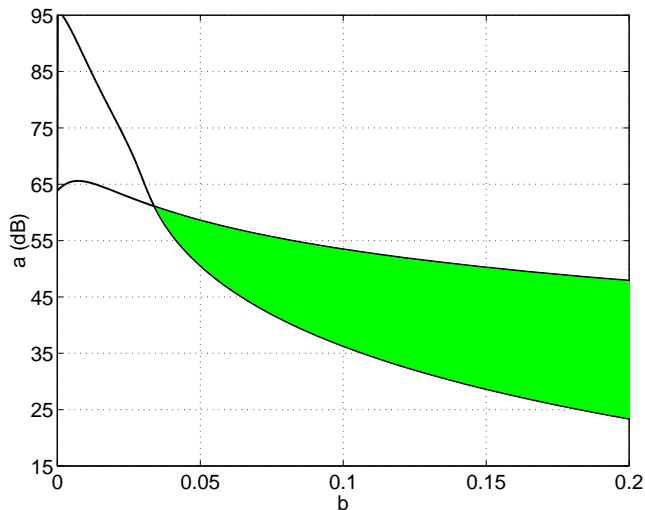


Figure 3: Region of (a, b) values for $M = 1.46$, equivalent to at least 40 deg phase margin and at least 10 dB gain margin ($K = 1$) for Example 2.

sign challenge here is to find all (a, b) pairs that solve the problem stated in Section 2 where $P(s)$ can be any member of the set. This (a, b) region will be the intersection of all (a, b) regions of members of the set (if this intersection region is empty, then there exists no PI solution). As an example, consider the plant set

$$P(s) = \frac{\text{gain}}{s(1 + s/\text{pole})}$$

where $\text{gain}=[1,3]$ and $\text{pole}=[10,12,14,16,18,20]$ and where $M = 1.46$ as before. Fig. 6 shows the intersection as the shaded region. The pair corresponding to maximum a is $(a, b) = (8.6 \text{ dB}, 0.62)$.

6 Discrete PI Controllers

The problem can be recast in its discrete form, where the plant is $P(z)$ and the controller (2) is replaced by its discrete equivalent,

$$C(z) = a_d (1 + b_d (1 - z^{-1})) .$$

Using the bilinear transformation,

$$z = \frac{1 + j\Omega}{1 - j\Omega}, \quad \Omega = j\omega T/2 ,$$

the plant can be written in the form $P((1 + j\Omega)/(1 - j\Omega))$, the controller in the form,

$$C(\Omega) = \frac{a_d(1 + (2b_d + 1)j\Omega)}{1 + j\Omega} ,$$

and the open-loop transfer function in the form,

$$L(\Omega) = a_d (1 + (2b_d + 1)j\Omega) \frac{P((1 + j\Omega)/(1 - j\Omega))}{1 + j\Omega} .$$

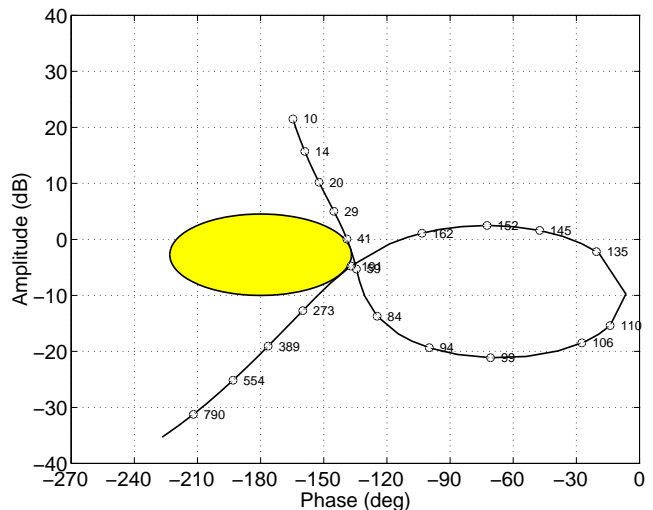


Figure 4: Nichols plot for $M = 1.46$ and $K = 1$, corresponding to at least 40 deg PM and at least 10 dB GM for Example 2. Frequencies are marked in rad/sec. The open-loop transfer function must not enter the shaded region in order to satisfy the M constraint.

The latter three equations translate the discrete problem into the one defined above for finding the (a_d, b_d) region. The procedure is as follows: Solve the problem defined in Section (2) where $P(z)$ at the frequencies on the unit circle are replaced by $P((1 + j\Omega)/(1 - j\Omega))/(1 + j\Omega)$ to determine the (a, b) region. Then (a_d, b_d) will be the region defined by $(a, (b - 1)/2)$.

7 Conclusions

The paper presents explicit equations for calculating PI controllers that simultaneously stabilize a given set of plants and satisfy design specifications, namely GM and PM constraints and a bound on the (complementary) sensitivity, for continuous as well as discrete-time systems. The algorithm fits any plant dimension including pure delay. Moreover, the algorithm answers the question if a solution whose bandwidth is in a given interval exists or not.

The two parameters of PI controllers satisfying the constraints correspond to a domain in a plane whose boundary is a curve given explicitly. For a practical optimization criterion presented here, the optimal controller lies on the curve. By inspection, the design plot enables identification of the PI controller for desired robustness conditions, and in particular, gives the PI controller for lowest sensitivity. Tradeoffs among high frequency sensor noise, low frequency sensitivity, and gain and phase margin constraints are also directly available.

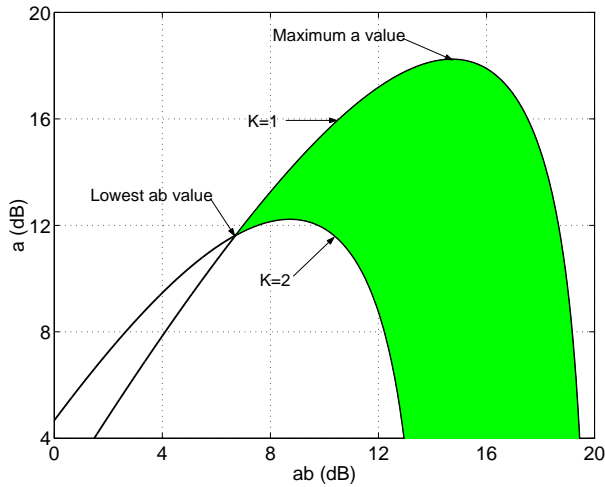


Figure 5: The (a, ab) region, shaded, satisfying $|\frac{1}{1+kL}| < 2$ where $k \in [1, 2]$ for Example 1.

The algorithm can be executed very fast for highly uncertain plants, and as such the controller design can be updated in near real-time to reflect changes in plant uncertainty and/or closed-loop specifications.

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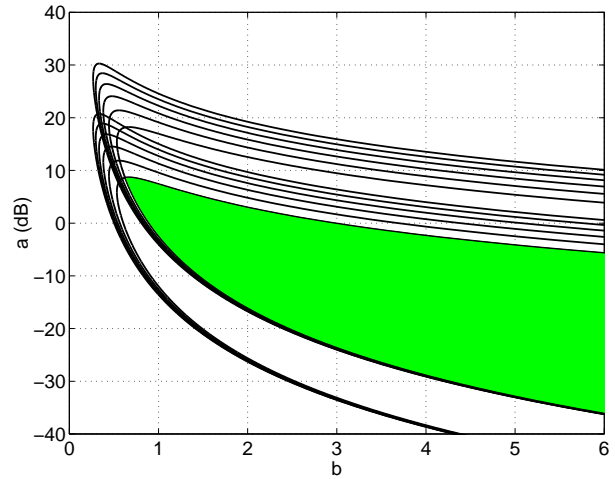


Figure 6: Boundary curves of (a, b) region that satisfy $|\frac{1}{1+L}| < M$ for $M = 1.46$ for a set of plants. The intersection of all regions is the allowed region. $M = 1.46$ is equivalent to 40 deg PM or greater and 10 dB GM or greater.

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