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TECHNICAL NOTES

AN INTERACTIVE GRAPHICS PACKAGE FOR CALCULATING CROSS-SECTIONAL PROPERTIES OF COMPLEX SHAPES*

INTRODUCTION

The cross-sectional properties of complex shapes are required in many studies of musculoskeletal biomechanics. To calculate these properties, many previous investigators have used regular geometries such as circles or ellipses to represent biological cross-sections. However, this approach involves a degree of simplification which often does not adequately reflect the complexity of the actual shape. As a result, the subtle changes in cross-sectional properties that can occur in adaptive biological systems have been inadequately studied. In particular, the mechanisms involved in the stress-related adaptive response of bone have remained obscure. The obvious limitations and inaccuracies of modeling irregular geometries by circles, ellipses, triangles and rectangles, all of known geometric properties, have led to more refined, but more painstaking and time-consuming, approaches. For instance, placing a grid over the cross-section and summing the subsequent enclosed blocks and noting their distribution has been used to estimate the area and area inertial properties (Lovejoy *et al.*, 1976; Uhthoff and Jaworski, 1978; Piotrowski and Kellman, 1978). In this method, greater accuracy requires more refined gridding, which in turn requires more work and time. In another approach, polar dot paper has been used in place of the gridded overlay, but the basic limitations of the point-counting method remain (Martin, 1975). In a refinement of the block-gridding technique, finite-element programs with automated mesh generation have also been used to calculate area inertial properties (Piziali *et al.*, 1976).

The purpose of this note is to provide an interactive computer graphics software package for calculating the area cross-sectional properties of irregular, two-dimensional shapes. The program requires input of the boundary coordinates of multiple, planar regions, each of which may be multiply connected. The program then calculates the area, centroid, area moments of inertia, and principal moments and their orientations. The approach thus allows rapid analysis of the section properties of complex biological cross-sections and facilitates both the mechanical analysis of musculoskeletal components and the quantification of biological changes in cross-sectional geometry due to aging, metabolic bone disease, or the presence of an internal fixation device. In the following, the basic algorithm used for these calculations is described and the FORTRAN implementation of the technique is outlined. Examples of the application of the method are then discussed.

ANALYTIC METHODS

The software package described in this note is based on a simple algorithm for generating area cross-sectional properties from perimeter coordinates (Wojciechowski, 1976; Hewlett Packard HP-97 ME Pac). The method divides a cross-sectional area into a series of trapezoids (or rectangles) and then adds or subtracts the properties of the elemental areas to determine the composite properties of the total area. This technique replaces integration with the summation of contributions of finite regions, and assumes linear segments between consecutive perimeter coordinates.

Consider the problem of calculating the area and area inertial properties of the shaded region in Fig. 1. This shaded region represents the region below the assumed linear segment connecting consecutive perimeter coordinates (G and E) of an entire cross-section boundary. To calculate the area under the straight line GE,

$$\Delta A = (X_{n+1} - X_n)(Y_{n+1} + Y_n)/2.$$

The static moment of ΔA (defined by trapezoid $ABEG$) about the x axis, ΔM_x, can be considered to consist of contributions from rectangle $ABCD$ and triangle FCE minus the contribution from rectangle GFD. Using the areas and centroids (from basic formulae) yields

$$\Delta M_x = [(X_{n+1} - X_n)/8] \times [(Y_{n+1} + Y_n)^2 + (Y_{n+1} - Y_n)^2/3].$$

Similarly, the moment of inertia ΔI_x of ΔA about the x axis consists of contributions from rectangle $ABCD$ and triangle FCE minus the contribution from triangle GFD. Existing formulae for the moments of inertia and the parallel axis theorem give

$$\Delta I_x = [(X_{n+1} - X_n)(Y_{n+1} + Y_n)/24] \times [(Y_{n+1} + Y_n)^2 + (Y_{n+1} - Y_n)^2].$$

The individual results are summed to arrive at the total area, A, the static moment, M_x, and the moment of inertia, I_x, of a plane cross-section:

$$A = \Sigma \Delta A \qquad M_x = \Sigma \Delta M_x \qquad I_x = \Sigma \Delta I_x.$$

The y centroid coordinate, $\bar{y}$, is

$$\bar{y} = M_x/A.$$

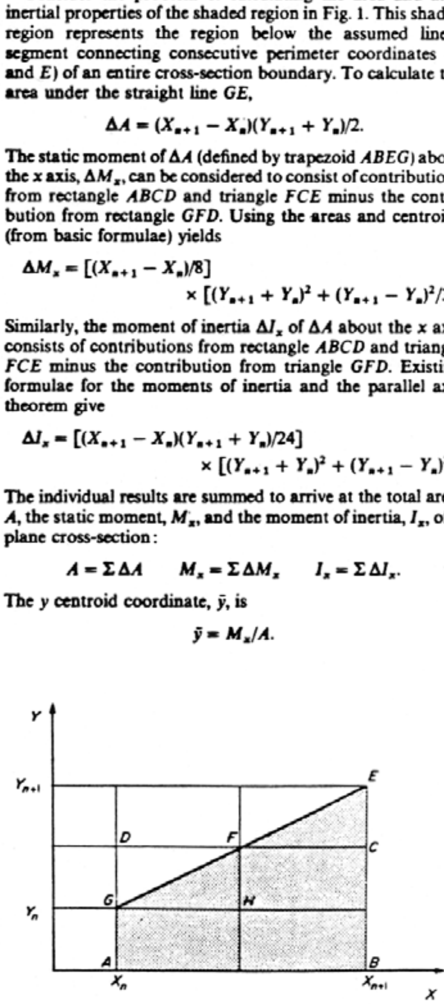

Fig. 1. Area section properties under line GE.

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The moment of inertia about the axis through $\bar{y}$ parallel to the x axis is

$$I_{\bar{x}} = I_x - \bar{y}^2 A.$$

Using the same approach, expressions can be derived for the moment of inertia about the y axis (I_y), the product of inertia (I_{xy}), the x coordinate of the centroid ($\bar{x}$), and the moment of inertia about the x axis translated to the centroid ($I_{\bar{y}}$). In addition, by standard formulae, the angle between the translated and principal axes (ϕ), and the moments of inertia about the translated and rotated principal x and y axes ($I_{\bar{x}\phi}$ and $I_{\bar{y}\phi}$, respectively) can be determined. These equations are presented in detail in the referenced Hewlett-Packard HP-97 ME Applications Pac.

PROGRAM SLICE

SLICE is the FORTRAN implementation of this method which was developed and executed on a DEC System-10 at the University of Pennsylvania Medical School Computer Facility. Using a Tektronix 4010 graphics terminal, the program graphically displays the cross-section with superimposed vectors which represent the principal moments of inertia and their orientation. On the same display, section properties are reported to four significant figures. The output variables (and FORTRAN names) are the area (AREA); the x and y coordinates of the centroid (XBAR, YBAR); the moments of inertia about the x and y axes (IX, IY); the product of inertia (IXY); the moments of inertia about the x and y axes translated to the centroid (IXBAR, IYBAR); the product of inertia about the translated axes (IXBYB); the angle between the translated axis and principal axis (PHI); and the moments of inertia about the translated and rotated principal x and y axes (IXBPH, IYBPH). Thus, the cross-section picture and section property information are output on one display "page" (screen), allowing for rapid correlation between changes in cross-section geometries and resulting changes in section properties.

Figure 2 shows the general programming flowchart of SLICE. The heart of the program is subroutine SLICER, which incorporates the algorithms described in the Analytic Methods section of this note. The FORTRAN listing of subroutine SLICER, complete with a list of variables, appears in the Appendix. SLICER must be passed: (1) the total number of perimeters for the cross-section; (2) the number of coordinate points for each perimeter (not necessarily equal for all perimeters); and (3) arrays of the x, y coordinates for each perimeter. As currently implemented, the cross-section may have up to a total of 20 boundary perimeters, each perimeter having a maximum of 500 coordinates, but the arrays can easily be redimensioned to accommodate cross-sections with more boundaries and/or with more coordinates.

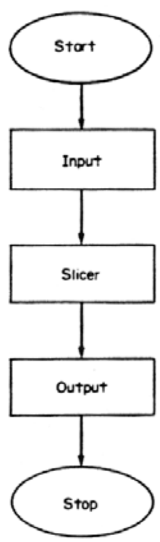

Fig. 2. SLICE flowchart.

To use SLICE, the cross-section must be located entirely within the first quadrant, that is, with all perimeter coordinates greater than or equal to zero. The cross-section may be multiply-connected, defined by an outer perimeter and inner perimeters, denoting "holes" or cross-section voids. Outer perimeter coordinates must be input sequentially in a clockwise path around the boundary. Coordinate points for each inner perimeter must be input in a counter-clockwise path. The program automatically closes a cross-section perimeter (for outer or inner perimeters of the cross-section) when the distance between a coordinate point and the first input point of the perimeter becomes less than a tolerance distance which is based on the first few input points. Once a perimeter is closed, the program ignores all subsequent input points until a point outside a circle of radius tolerance distance is found. By this means, the program can distinguish between cross-section perimeters after reading in a continuous string of coordinates of the boundaries. SLICE accepts boundary coordinates of the outer and interior perimeters by direct terminal keyboard input or from data files. However, the program can easily be adapted to accept x, y coordinates directly from a table digitizer.

APPLICATIONS

A number of simple geometric shapes of known area properties have been used as test examples to validate the program and to verify its high degree of accuracy. With simple geometric shapes, such as rectangles and triangles, the program attains accuracy to 7 significant figures. For execution of both simple test examples and more complex bone cross-sections, the program requires approximately 25 sec of CPU time (at a cost of just over 1 dollar). Total cost per cross-section run is relatively constant at approximately 3 dollars with complete graphics output as shown in the following examples.

A simple example of SLICE graphics output appears in Fig. 3. It represents a triangular cross-section with a triangular "hole". Note that the principal directions are represented by vectors superimposed at the centroid, with their lengths proportional to the principal moments of inertia. In this case, the moment of inertia about the principal y axis (IYBPH) is more than 5 times larger than the moment of inertia about the principal x axis (IXBPH).

SLICE has been used as a basic subroutine in stress analysis programs which calculate stresses based on elementary beam theory for pure bending and torsion. Figure 4 shows SLICE output for a cross-section of an orthopaedic bone reamer which had failed in clinical use. (Note that the reamer cross-section is a regular, albeit complicated, geometric shape. It would be possible—but painstaking—to calculate its exact area properties by hand by considering the cross-section as a combination of simple geometric shapes such as triangles.) In this case, the stress analyses provide both a reasonable explanation of the reamer's failure and a simple means for investigating design changes.

SLICE has been incorporated as a subroutine in mass properties programs, which calculate the mass and mass inertial properties of three-dimensional bodies (with any number of different density interior sections). These programs have been coupled with computerized axial tomography (CAT) scanning systems, which non-invasively and directly construct cross-sectional pictures and mass densities at multiple intervals along a limb or body segment, to directly generate mass and mass inertial properties.

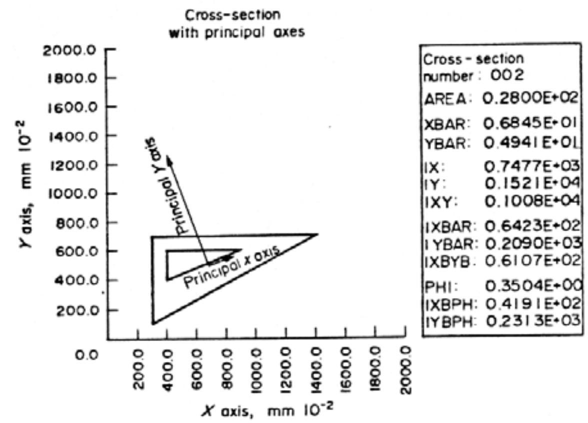

Fig. 3. Test example for program verification. Perimeter coordinates for outer triangle are (3, 1), (3, 7), (14, 7). Inner triangle coordinates are (4, 4), (9,6), (4, 6).

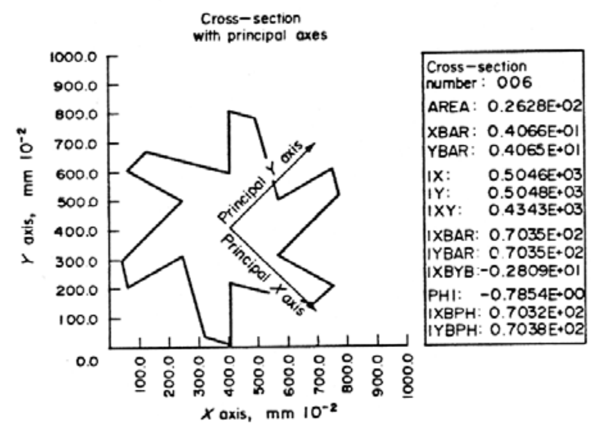

Fig. 4. Distal cross-section of orthopaedic reamer.

Cross-sectional changes in bones subjected to different environmental conditions have also been studied quantitatively with the program. Using microradiographs provided by Slatis *et al.* (1978), cross-sections of rabbit tibias examined 1 day–36 weeks after compression plate implantation were input to the program to monitor cross-sectional changes and to optimize placement of the plate (Fig. 5). Results of this analysis will be reported separately. They emphasize, however, the advantages of an automated method for computing the cross-sectional properties of musculoskeletal structures.

In summary, a simple algorithm for computing the cross-sectional properties of complex geometric shapes has been programmed for automatic data acquisition and analysis. The program can be used to analyze the area properties of multiple, complex biological cross-sections. Specifically, biological changes in cross-sectional geometries can be quantified to better understand changes due to aging, metabolic bone disease, and the presence of fracture fixation devices. Since the approach is highly automated and makes use of extensive graphics output, area property changes can easily be investigated. The program has been completely verified using known cross-sectional geometries and represents an advance over previous methods in ease of use, time, cost and accuracy.

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Department of Orthopaedic Surgical Research,
424 Medical Education Building,
University of Pennsylvania,
Philadelphia, PA 19104, U.S.A.

Mark L. Nagurka*
Wilson C. Hayes

* Address for reprints: Orthopaedic Biomechanics laboratory, Beth Israel Hospital, 330 Brookline Avenue, Boston, MA 02215, U.S.A.

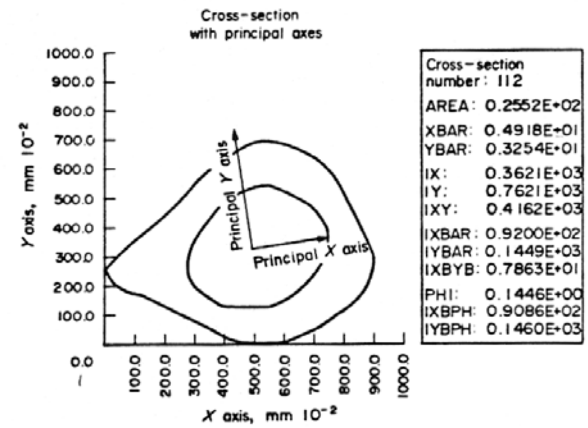

Fig. 5. Cross-section of rabbit tibia after removal of compression plate.

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APPENDIX

Subroutine SLICER

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C
C
C      * * * * * * * *  SLICE  * * * * * * * * * *
C
C
C
C     AUTHOR :                     MARK L. NAGUEKA
C     PRINCIPAL INVESTIGATOR:  WILSON C. HAYES, PHD.
C
C        DEPARTMENT OF ORTHOPAEDIC SURGICAL RESEARCH
C        UNIVERSITY OF PENNSYLVANIA
C        PHILADELPHIA, PENNSYLVANIA  19104
C
C
C        THIS WORK WAS SUPPORTED BY GRANT AM 20135-02  (MECHANICS
C        OF NORMAL, ARTHRITIC, AND PROSTHETIC KNEES) FROM THE NATIONAL
C        INSTITUTES OF HEALTH.
C
C
C
C
C        PROGRAM SLICE CALCULATES AREA AND AREA INERTIAL PROPERTIES
C        OF ANY CROSS-SECTION CONTAINING ANY NUMBER OF INTERIOR
C        SECTIONS (IE. HOLES) GIVEN X,Y COORDINATE POINTS OF OUTER
C        AND INTERIOR PERIMETERS, EITHER BY MANUAL DATA INPUT
C        OR DATA FILE, AND DISPLAYS CROSS-SECTION GRAPHICALLY.
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C       * * * * VARIABLES * * * *
C
C       AREA - CROSS-SECTION AREA
C       I - X,Y COORDINATE INDEX
C       IX - MOMENT OF INERTIA ABOUT X-AXIS
C       IXBAR - MOMENT OF INERTIA ABOUT X-AXIS TRANSLATED TO CENTROID
C       IXBPH - MOMENT OF INERTIA ABOUT TRANSLATED, ROTATED,
C          PRINCIPAL X-AXIS
C       IXPRM - X MOMENT OF INERTIA ABOUT ARBITRARY AXIS
C       IXY - PRODUCT OF INERTIA
C       IXYPR - PRODUCT OF INERTIA ABOUT ARBITRARY AXIS
C       IY - MOMENT OF INERTIA ABOUT Y-AXIS
C       IYBAR - MOMENT OF INERTIA ABOUT Y-AXIS TRANSLATED TO CENTROID
C       IYBPH - MOMENT OF INERTIA ABOUT TRANSLATED, ROTATED
C          PRINCIPAL Y-AXIS
C       IYPRM - Y MOMENT OF INERTIA ABOUT ARBITRARY AXIS
C       J - POLAR MOMENT OF INERTIA ABOUT ARBITRARY AXIS
C       NPER - NUMBER OF PERIMETERS
C       NPOINT(IPER) - NUMBER OF POINTS IN SECTION (IPER)
C       PHI - ANGLE BETWEEN TRANSLATED AXIS AND PRINCIPAL AXIS
C       THETA - ANGLE BETWEEN ORIGINAL AXIS AND ARBITRARY AXIS
C       X(I) - X COORDINATE OF PREVIOUS VERTEX POINT
C       X(I+1) - X COORDINATE OF CURRENT VERTEX POINT
C       XBAR - X COORDINATE OF CENTROID
C       XBARA - X COORDINTE OF CENTROID TIMES AREA
C       XVECT(IPER,I) - ARRAY OF X(I) FOR SECTION IPER
C       Y(I) - Y COORDINATE OF PREVIOUS VERTEX POINT
C       Y(I+1) - Y COORDINATE OF CURRENT VERTEX POINT
C       YBAR - Y COORDINATE OF CENTROID
C       YBARA - Y COORDINATE OF CENTROID TIMES AREA
C       YVECT(IPER,I) - ARRAY OF Y(I) FOR SECTION IPER
C


C
C       ...........................................................


C       SUBROUTINE SLICER REDUCES SECTION PERIMETER INFO
C       TO AREA AND AREA INERTIAL PROPERTIES OF SECTION.


        SUBROUTINE SLICER

        COMMON/IN/XVECT(20,500),YVECT(20,500),NPOINT(20),NPER,
       1THETA
        COMMON/OUT/AREA,XBAR,YBAR,IX,IY,IXY,IXBAR,IYBAR,
       1IXBYB,PHI,IXBPH,IYBPH,IXPRM,IYPRM,J,IXYPR

        DIMENSION X(500),Y(500)
        REAL MAG,IX,IY,IXY,IXBAR,IYBAR,IXBYB,PHI,IXPPH,
       1IYBPH,IXPRM,IYPRM,IXYPR,J,IDENT

C       INITIALIZE
        AREA = 0.0
        XBARA = 0.0
        YBARA = 0.0
        IX = 0.0
        IY = 0.0
        IXY = 0.0

        IF(NPER.EQ.1) IPER=1
        IF(NPER.EQ.1) GOTO 10
        DO 50 IPER=1,NPER

C       FILL X,Y VECTORS FOR EACH SECTION PERIMETER
10      N1=NPOINT(IPER)+1
        DO 20 I=1,N1
        X(I)=XVECT(IPER,I)
        Y(I)=YVECT(IPER,I)
20      CONTINUE

C       CALCULATIONS
        DO 40 I=1,NPOINT(IPER)
        AREA=AREA-((Y(I+1)-Y(I))*(X(I+1)+X(I)))/2
        XBARA=XBARA-(((Y(I+1)-Y(I))/8)*((X(I+1)+X(I))**2
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     1+((X(I+1)-X(I))**2)/3))
      YBARA=YBARA+(((X(I+1)-X(1))/8)*((Y(I+1)+Y(I))**2
     1+((Y(I+1)-Y(1))**2)/3))
      IX=IX+((X(I+1)-X(I))*(Y(I+1)+Y(I))/24)*
     1((Y(I+1)+Y(1))**2+(Y(I+1)-Y(1))**2)
      IY=IY-((Y(I+1)-Y(I))*(X(I+1)+X(I))/24)*
     1((X(I+1)+X(I))**2+(X(I+1)-X(1))**2)
      IF((X(I+1)-X(I)).EQ.0) GOTO 40
      IXY=IXY+(((Y(I+1)-Y(I))**2)*(X(I+1)+X(I))*
     1((X(I+1))**2+(X(I))**2)/8+
     2(Y(I+1)-Y(I))*(X(I+1)*Y(I)-X(I)*Y(I+1))*
     3((X(I+1))**2+X(I+1)*X(I)+(X(I))**2)/3+
     4((X(I+1)*Y(I)-X(I)*Y(I+1))**2)*
     5(X(I+1)+X(I))/4)/(X(1+1)-X(1))
40    CONTINUE
50    CONTINUE


C     EXTRA CALCULATIONS
      XBAR=XBARA/AREA
      YBAR=YBARA/AREA
      IXBAR=IX-AREA*YBAR**2
      IYBAR=IY-AREA*XBAR**2
      IXBYB=IXY-AREA*XBAR*YBAR
      BIG=IXBAR
      IF(IYBAR.GT.IXBAR) BIG=IYBAR
      IF((ABS(IXBAR-IYBAR)/BIG).LE.1E-5) GOTO 60
      PHI=(ATAN(-2*IXBYB/(IXBAR-IYBAR)))/2
      GOTO 70
60    PHI = -0.78539816
70    IXBPH=IXBAR*(COS(PHI))**2+IYBAR*(SIN(PHI))**2-
     1IXBYB*SIN(2*PHI)
      IYBPH=IYBAR*(COS(PHI))**2+IXBAR*(SIN(PHI))**2+
     1IXBYB*SIN(2*PHI)
C
C     CALCULATIONS ABOUT ARBITRARY AXIS
      IF(THETA.EQ.0.0) GOTO 80

      IXPRM=IXBAR*(COS(THETA))**2+IYBAR*(SIN(THETA))**2
     1-IXBYB*SIN(2*THETA)
      IYPRM=IYBAR*(COS(THETA))**2+IXBAR*(SIN(THETA))**2
     1+IXBYB*SIN(2*THETA)
      J=IXPRM+IYPRM
      IXYPR=(SIN(2*THETA))*(IXBAR-IYBAR)/2+IXBYB*(COS(2*THETA
     1))

80    RETURN
      END
```