A Mass-Spring-Damper Model of a Bouncing Ball

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***Abstract*— The mechanical properties of a vertically dropped ball, represented by an equivalent mass-spring-damper model, are related to the coefficient of restitution and the time of contact of the ball during one bounce with the impacting surface. In addition, it is shown that the coefficient of restitution and contact time of a single bounce are related to the total number of bounces and the total time elapsing between dropping the ball and the ball coming to rest. For a ball with significant bounce, approximate expressions for model parameters, i.e., stiffness and damping or equivalently natural frequency and damping ratio, are developed. Experimentally-based results for a bouncing ping-pong ball are presented.**

I. Introduction

The bouncing behavior of a dropped ball is a classic problem studied in depth [1]-[5]. The topic is treated in virtually all textbooks of physics and dynamics that address the subject of impact. These books also present, but in a separate section, the concept of mass, stiffness, and damping as the three elemental properties of a mechanical system. To the authors' knowledge, the textbooks and references do not make a connection between the mechanical "primitives" of mass, stiffness and damping and the coefficient of restitution, presented as part of the subject of impact. This paper develops this connection for a particular system, namely a bouncing ball, represented by a linear mass-spring-damper model. It is shown that the properties of the ball model can be related to the coefficient of restitution and bounce contact time. Furthermore, for the vertically dropped ball problem it is shown that the total number of bounces and the total bounce time, two parameters that are readily available experimentally, can be related to the stiffness and damping. The analytical findings are tested to predict model properties of a ping-pong ball.

II. Mass-Spring-Damper Model

To study the behavior of a vertically dropped ball, consider the model illustrated in Figure 1, where the ball is represented by its mass m, viscous damping c, and linear stiffness k. When the ball is not in contact with the ground, the equation of motion, assuming no aerodynamic drag, can be written simply as

$$m\ddot{x} = -mg \; , \tag{1}$$

where x is measured vertically up to the ball's center of mass with $x = 0$ corresponding to initial contact, i.e., when the ball just contacts the ground with no deformation. The initial conditions are $x(0) = h_0$ and $\dot{x}(0) = 0$ for a ball released from rest from height h_0. The solution of this simple problem appears in physics and mechanics textbooks, leading to the classical results of vertical projectile motion.

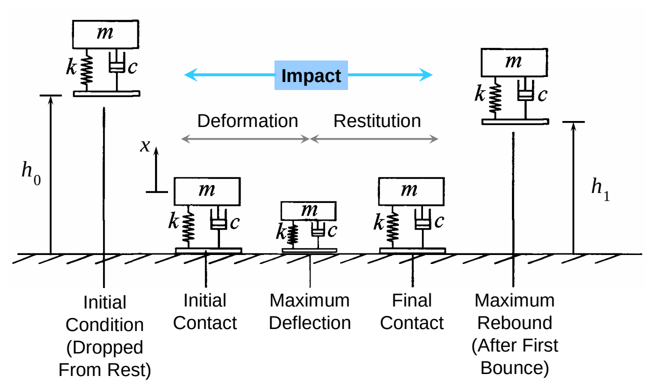

Fig. 1. A mass-spring-damper model of a ball showing phases in impact at first bounce.

When the ball is in contact with the ground, deformation and restitution occur. The equation of motion is then,

$$m\ddot{x} + c\dot{x} + kx = -mg \tag{2}$$

with the initial conditions of $x(0) = 0$ and $\dot{x}(0) = -v_0$ where v_0 is the velocity of the ball just prior to contact with the ground. Integrating eq. (2) gives

$$x = \left[\frac{cg - 2kv_0}{2k\omega_d}\sin\omega_d t + \frac{mg}{k}\cos\omega_d t\right] \times \exp\left(-\frac{c}{2m}t\right) - \frac{mg}{k} \tag{3}$$

where the damped natural frequency, ω_d, is

$$\omega_d = \frac{1}{2m}\sqrt{4km - c^2} \; . \tag{4}$$

Equation (3) gives the motion of the ball during contact with the ground and applies only when $x \leq 0$. Bounce behavior, involving deformation, restitution, and then rebound, requires an underdamped solution for which $\omega_d > 0$ or $(4km - c^2) > 0$.

The "steady" or rest solution, applying after the bounces have died out, can be obtained by setting $t \to \infty$ in eq. (3). The equilibrium position is

$$x^* = -\frac{mg}{k} \; , \tag{5}$$

and when $|x| \leq |x^*|$ there will be no further bounces. It follows that the number of bounces is finite.

A. Time of Contact

The time of contact, ΔT, for the first bounce, shown in exaggerated view in Figure 2, is the time from when the ball reaches $x = 0$ after being dropped to the time it first comes back to $x = 0$. Mathematically, the contact time is the first finite solution of the equation $x(\Delta T) = 0$, i.e., it is the minimum non-zero solution of

$$\left[\frac{cg - 2kv_0}{2k\omega_d}\sin(\omega_d \Delta T) + \frac{mg}{k}\cos(\omega_d \Delta T)\right] \times \exp\left(-\frac{c\Delta T}{2m}\right) - \frac{mg}{k} = 0 \; , \qquad (6)$$

which in general has multiple solutions.

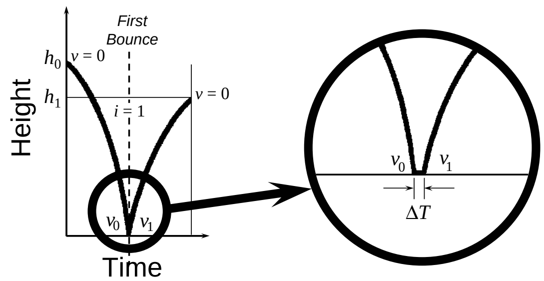

Fig. 2. Height versus time and exaggerated view at first bounce.

Although eq. (6) is difficult to solve analytically, it can be solved numerically. Alternatively, an approximate solution can be obtained. Start by writing eq. (3) in the rearranged form,

$$x = -\frac{v_0}{\omega_d}\exp\left(-\frac{c}{2m}t\right)\sin\omega_d t + \frac{mg}{k} \times \left[\exp\left(-\frac{c}{2m}t\right)\left(\cos\omega_d t + \frac{c}{2m\omega}\sin\omega_d t\right) - 1\right] \; . \qquad (7)$$

Assuming $\frac{mg}{k} \ll 1$, which is reasonable for a bouncing ball such as a ping-pong ball, the second term on the right-hand side in (7) can be neglected and x can be approximated as

$$x = -\frac{v_0}{\omega_d}\exp\left(-\frac{c}{2m}t\right)\sin\omega_d t \; . \qquad (8)$$

The contact time, ΔT, can be found as the minimum nonzero solution of eq. (8) set equal to zero giving

$$\Delta T = \frac{\pi}{\omega_d} \; , \qquad (9)$$

where ω_d is specified by eq. (4). Equation (9) represents an approximate solution for the contact time at the first bounce.

B. Stiffness and Damping

The ball properties, k and c, can be determined from the contact time, ΔT, and the coefficient of restitution, e, where

$$e = \left|\frac{\dot{x}(\Delta T)}{\dot{x}(0)}\right| \; . \qquad (10)$$

The denominator of eq. (10) is simply the velocity of the ball prior to contact, v_0, and the numerator is the rebound or post-impact velocity of the ball, v_1. The latter can be found by differentiating eq. (7) and imposing the assumption $\frac{mg}{k} \ll 1$ or alternatively differentiating eq. (8) directly to give an expression for the velocity,

$$\dot{x} = \frac{cv_0}{2m\omega_d}\exp\left(-\frac{c}{2m}t\right)\sin\omega_d t - v_0\exp\left(-\frac{c}{2m}t\right)\cos\omega_d t \; , \qquad (11)$$

and then substituting $t = \Delta T$ with eq. (9) to give the rebound velocity,

$$v_1 = \dot{x}(\Delta T) = v_0 \exp\left(-\frac{c\pi}{2m\omega_d}\right) \; . \qquad (12)$$

Thus, from eq. (10), the coefficient of restitution can be written simply as

$$e = \exp\left(-\frac{c\pi}{2m\omega_d}\right) \; . \qquad (13)$$

By manipulating eqs. (4), (9), and (13), the stiffness and viscous damping can be written, respectively, as,

$$k = \frac{m}{(\Delta T)^2}\left[\pi^2 + (\ln e)^2\right] \qquad (14)$$

$$c = -\frac{2m}{\Delta T}\ln e \; . \qquad (15)$$

Assuming k, c and e are constant (independent of the velocity v_0), ΔT will be constant for each contact since ω_d depends only on the system parameters k, c and m.

C. Natural Frequency and Damping Ratio

The undamped natural frequency, $\omega_n = \sqrt{k/m}$, can be expressed from eq. (14) as

$$\omega_n = \frac{1}{\Delta T}\sqrt{[\pi^2 + (\ln e)^2]} \qquad (16)$$

The damping ratio, ζ,

$$\zeta = \sqrt{1 - \frac{{\omega_d}^2}{{\omega_n}^2}} = \frac{c}{2\sqrt{km}}$$

can be found by substituting eqs. (14) and (15) giving

$$\zeta = -\frac{\ln e}{\sqrt{\pi^2 + (\ln e)^2}} \qquad (17)$$

Eq. (17) indicates that the damping ratio depends solely on the coefficient of restitution.

D. Coefficient of Restitution and Time of Contact

For a given ball, the mass m is readily available whereas the parameters k and c or, alternatively, ω_n and ζ are generally unknown. From ΔT and e, which are also unknown (but can be found experimentally), k and c can be determined from eqs. (14) and (15), or ω_n and ζ can be determined from eqs. (16) and (17).

The total number of bounces of the ball, n, and the total time, T_{total}, that elapses from when the ball is dropped until it comes to rest are two parameters that can be determined readily in an experiment. They are indicated in the bounce history diagram of Figure 3. In the following, it is shown that with n and T_{total} assumed known, ΔT and e, and thus k and c, can be determined under the assumption of constant ΔT and e for all bounces and neglecting aerodynamic drag.

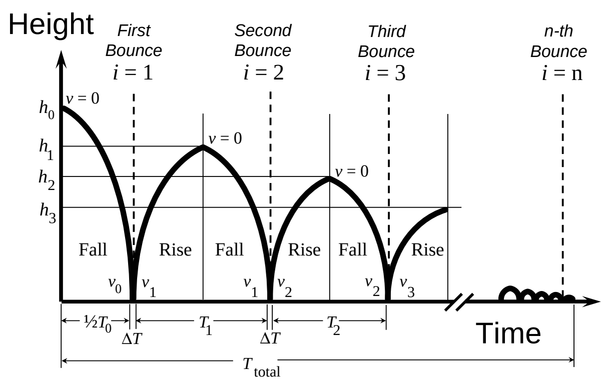

Fig. 3. Bounce history showing height versus time.

For the i-th bounce, the height the ball can reach is

$$h_i = e^{2i} h_0 \tag{18}$$

where h_0 is the height when the ball is dropped since $v_i = ev_{i-1} = e^i v_0$ and $v_i = \sqrt{2gh_i}$. For the ball to come to rest,

$$h_n = e^{2n} h_0 \leq \frac{mg}{k} \tag{19}$$

where the upper limit is given by the equilibrium position of (5). Substituting eq. (14) into the equality of (19) and rearranging gives an expression for the contact time, ΔT, in terms of unknown e:

$$\Delta T = e^n \sqrt{\frac{h_0}{g} \left[\pi^2 + (\ln e)^2\right]} \tag{20}$$

The total time is the sum of the total flight time, T_{flight}, and the total contact time, $T_{contact}$,

$$T_{total} = T_{flight} + T_{contact} \tag{21}$$

where

$$T_{flight} = \frac{1}{2} T_0 + \sum_{i=1}^{n} T_i \tag{22}$$

and

$$T_{contact} = n \Delta T \tag{23}$$

assuming the contact times at the bounces are identical. Noting that the flight time for the ith bounce can be written as $T_i = eT_{i-1}$ for $i \geq 2$ and $T_1 = eT_0$, the total flight time for the number of bounces n can be calculated using eq. (22):

$$\begin{aligned} T_{flight} &= \frac{1}{2} T_0 + T_1 + \cdots + T_n \\ &= \frac{1}{2} T_0 + T_1 (1 + e + \cdots + e^{n-1}) \\ &= \frac{1}{2} T_0 + T_0 e \left(\frac{1 - e^{n-1}}{1 - e} \right) \\ &= \frac{1}{2} T_0 \left(\frac{1 + e - 2e^n}{1 - e} \right) . \end{aligned}$$

Since $T_0 = 2\sqrt{\frac{2h_0}{g}}$, the total flight time for the number of bounces n can be expressed as,

$$T_{flight} = \sqrt{\frac{2h_0}{g}} \left(\frac{1 + e - 2e^n}{1 - e} \right) . \tag{24}$$

Substituting eqs. (20), (23), and (24) into (21) gives

$$T_{total} = \sqrt{\frac{h_0}{g}} \times \left[\sqrt{2} \left(\frac{1 + e - 2e^n}{1 - e} \right) + ne^n \sqrt{\pi^2 + (\ln e)^2} \right] . \tag{25}$$

Eq.(25) can be viewed as a single equation for unknown e in terms of T_{total}, n, and h_0. The latter three quantities can readily be determined experimentally.

E. Approximations

It is possible to develop simplified approximate relationships for the case of $|(\ln e)/\pi| \ll 1$, which for a ratio of 0.1 or smaller corresponds to $0.73 < e < 1$. This case would be representative of a ball with significant bounce, such as a ping-pong ball.

For this case, eq. (14) can be approximated as

$$k \cong m \left(\frac{\pi}{\Delta T} \right)^2 , \tag{26}$$

which itself is an approximation of eq. (9),

$$\Delta T \cong \frac{\pi}{\omega_n} , \tag{27}$$

i.e., the contact time at a single bounce is simply π times the inverse of the undamped natural frequency. The contact time can also be approximated, from eq. (20), as

$$\Delta T \cong e^n \pi \sqrt{\frac{h_0}{g}} . \tag{28}$$

From eq. (17), it is also possible to write the damping ratio for the case of higher values of e as

$$\zeta \cong -\frac{\ln e}{\pi} \tag{29}$$

providing a simple direct connection between the damping ratio and the coefficient of restitution.

Simplification of eq. (25) gives

$$T_{total} \cong \sqrt{\frac{2h_0}{g}} \left(\frac{1+e}{1-e} \right) \tag{30}$$

for larger n and e. Eq. (30) does not depend on n, and can be rearranged to find a simple equation for e in terms of T_{total}.

III. Numerical and Experimental Studies

A ping-pong ball (Harvard, one-star) was dropped from rest from a measured initial height of 30.5 cm onto a (butcher-block top) laboratory bench. The acoustic signals accompanying the ball-table impacts were recorded using a microphone attached to the sound card of a PC. The method follows the procedure described in [6].

From the temporal history of the bounce sounds of successive impacts, the total number of bounces was determined to be $n = 70$ and the total bounce time was determined to be $T_{total} = 7.5$ s.

The mass of the ball used in the experiment was measured to be $m = 2.50$g. (The ball used was an older official ball. The rules of the International Table Tennis Federation were changed in September 2000 and now mandate a 2.7 g ball.)

In addition to the acoustic measurement, a high-speed digital video (using a Redlake Imaging MotionScope) was taken.

A. Predicted Coefficient of Restitution

The coefficient of restitution can be found from eq. (25) given known initial height h_0, total time T_{total}, and number of bounces, n. The relationship is shown in Figure 4 for the case of $h_0 = 30.5$ cm and indicates that the total time is not significantly dependent on the number of bounces, especially for a large number of bounces.

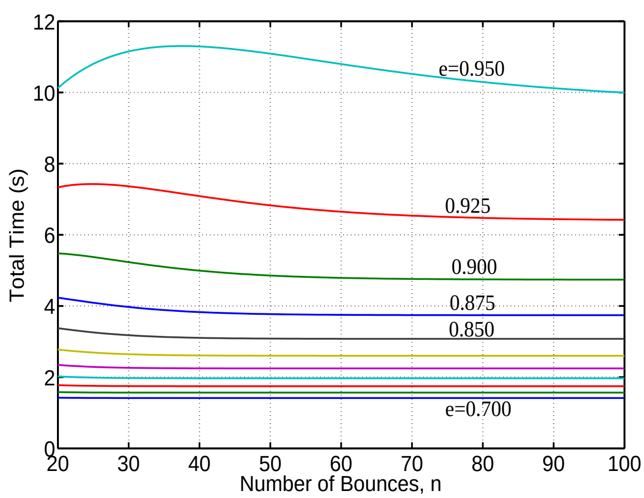

Fig. 4. Total time as a function of number of bounces and coefficient of restitution from eq. (25) for a drop height of 30.5 cm.

Figure 4 provides a means to identify by inspection the coefficient of restitution. In particular, for $T_{total} = 7.5$ s and $n = 70$, the coefficient of restitution is $e = 0.93$. This value is slightly higher than that determined for e at the first bounce based on pre- and post-impact velocities from the high-speed digital video images (i.e., by applying eq. (10)).

It is also possible to determine the coefficient of restitution from the approximate equation (30). From this equation, for $T_{total} = 7.5$ s, the coefficient of restitution is $e = 0.94$.

B. Predicted Contact Time

The contact time at a single bounce can be found from eq. (20) or the approximation of eq. (28). These relations are shown graphically in Figure 5, from which the contact can be determined by inspection given the total number of bounces, n, and the coefficient of restitution, e. For $n = 70$ and $e = 0.93$, the predicted contact time $\Delta T = 3.4$ ms. This value exceeds the contact time of $\Delta T = 1.0$ ms for the first bounce measured by the high-speed digital video system.

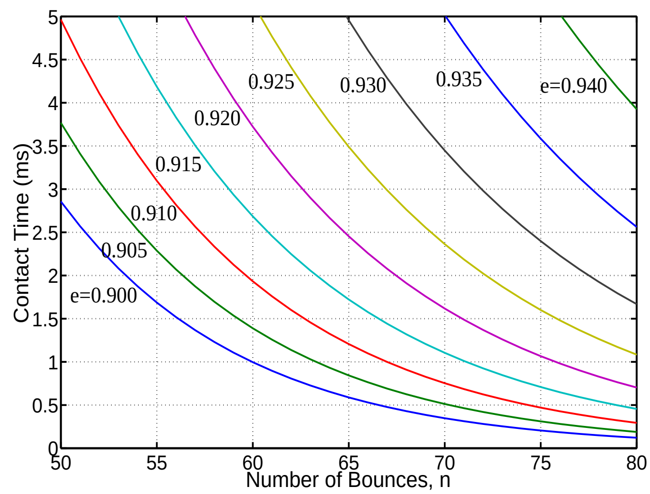

Fig. 5. Contact time at a single bounce as a function of number of bounces and coefficient of restitution from eq. (20) and from approximation of eq. (28) for a drop height of 30.5 cm.

C. Predicted Stiffness and Damping

Values of the linear stiffness and the viscous damping coefficient of an equivalent mass-spring-damper model of a ball can be determined.

An expression for the stiffness is given in eq. (14) and in simplified approximate form in eq. (26). Figure 6 graphically depicts these relationships in terms of k/m for the range of coefficient of restitution $0.40 \leq e \leq 0.95$ for several values of contact time. The approximate equation (26) provides a highly accurate prediction of the result from eq. (14), showing only slight deviation at smaller values of e.

For the case of the ping-pong ball dropped from an initial height of 30.5 cm and with ΔT determined to be 3.4 ms, $k/m = 8.5 \times 10^5$ s^2 and is not a function of e. For $m = 2.5$g, then the stiffness $k = 2.1$ N/mm (or kPa). As indicated

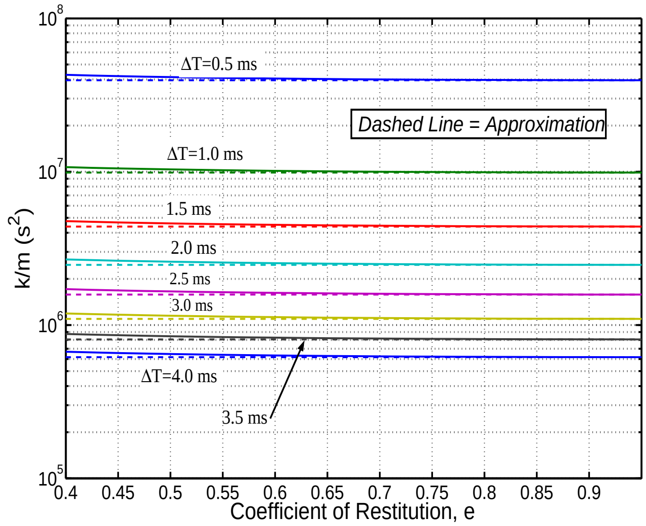

Fig. 6. Stiffness divided by mass as a function of coefficient of restitution and contact time from eq. (14) and for approximation from eq. (26).

above, ΔT was measured to be 1.0 ms from the high-speed digital video. With this value, $k/m = 1.0 \times 10^7$ s^2 and the stiffness $k = 25$ N/mm (or kPa).

An equation for the damping coefficient c was developed in eq. (15), and is plotted in Figure 7 as c/m as a function of both e and ΔT, showing clear dependence on both.

For $e = 0.93$ and $\Delta T = 3.4$ ms, $c/m = 43$ s^{-1} and for $m = 2.5$g then the damping coefficient $c = 0.11$ N·s/m. For the case of $e = 0.93$ and $\Delta T = 1.0$ ms, $c/m = 145$ s^{-1} and $c = 0.36$ N·s/m. It is noted that the equivalent damping is predicated on knowledge of e and the value of the contact time ΔT.

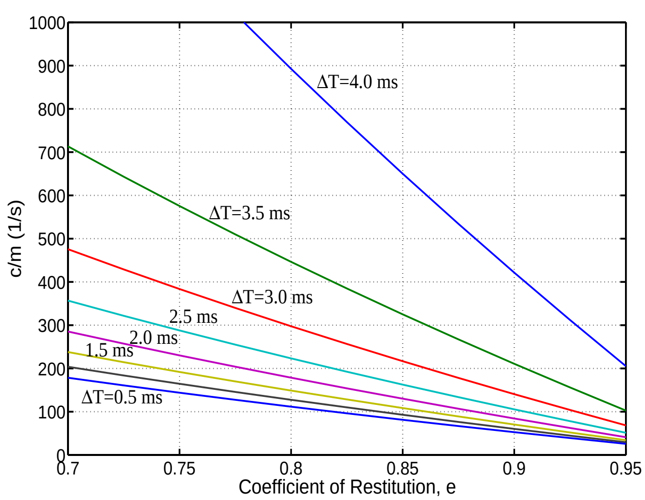

Fig. 7. Damping coefficient divided by mass as a function of coefficient of restitution and contact time from eq. (15).

D. Predicted Natural Frequency and Damping Ratio

From eq. (16) or the approximation from a rearrangement of eq. (27) it is possible to find the natural frequency. For $\Delta T = 3.4$ ms and $e = 0.93$, $\omega_n = 920$ rad/s or 150 Hz. For the case of $\Delta T = 1.0$ ms and $e = 0.93$, $\omega_n = 3100$ rad/s or 500 Hz.

From eq. (17) or the approximation of eq. (29) it is possible to predict the damping ratio. For $e = 0.93$, the damping ratio $\zeta = 0.023$. The small value of damping ratio indicates a very lightly underdamped system.

IV. Discussion

The total time from the when the ball is dropped until when it comes to rest is comprised of two phases: flight times and contact times. Although the total contact time summed for all bounces is a small fraction of the total flight time (approximately 3 percent), it is included in the model.

The analytical development assumes constant mass-spring-damper model parameters, m, k, and c, and constant coefficient of restitution, e. A consequence of assuming that these parameters are constant is a constant contact time, ΔT, at each bounce.

The analysis neglects aerodynamic effects, which occur in reality. By not accounting for aerodynamic drag of the ball during flight, the approach gives a higher coefficient of restitution than otherwise would be predicted.

The approach predicts a contact time three times greater than that found by an independent method (3.4 ms vs 1.0 ms using high-speed digital video). Reconciling this large difference requires further study into the errors resulting from the underlying assumptions, namely, neglecting aerodynamic drag and adopting a linear, fixed mass-spring-damper model.

Several observations can be made: (*i*) the larger the contact time ΔT, the smaller the stiffness k and the larger the damping c, *(ii)* the larger the coefficient of restitution e, the smaller the damping c, *(iii)* the coefficient of restitution e does not strongly influence the stiffness k, *(iv)* the larger the coefficient of restitution e, the larger the total time, and *(v)* the number of bounces n (assuming $n >$ 20) does not strongly influence the total time.

V. Closing

This paper examines relationships bridging linear equivalent model parameters, namely the mass, stiffness, and damping of a bouncing ball, with the classical concept of coefficient of restitution and time of contact between a ball and a surface. Under the assumption of no aerodynamic drag and constant coefficient of restitution for all bounces, the derivation shows that the stiffness and damping, or alternatively the natural frequency and damping ratio, can be expressed explicitly in terms of the coefficient of restitution and time of contact. The formulation also considers the special case for bouncing balls involving higher values of the coefficient of restitution for which simple approximate expressions can be derived for parameters of the ball model. The results of an experimental test are used to provide predictions of the equivalent stiffness and damping, as well as natural frequency and damping ratio, and coefficient of restitution for a bouncing ping-pong ball.

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