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A New Geometric Perspective on MIMO Systems

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This paper promotes a new graphical representation of the behavior of linear, time-invariant, multivariable systems highly suited for exploring the influence of closed-loop system parameters. The development is based on the adjustment of a scalar control gain cascaded with a square multivariable plant embedded in an output feedback configuration. By tracking the closed–loop eigenvalues as an explicit function of gain, it is possible to visualize the multivariable root loci in a set of "gain plots" consisting of two graphs: (i) magnitude of system eigenvalues vs. gain, and (ii) argument (angle) of system eigenvalues vs. gain. By depicting unambiguously the polar coordinates of each eigenvalue in the complex plane, the gain plots complement the standard multi-input, multi-output root locus plot. Two example problems demonstrate the utility of gain plots for interpreting linear multivariable system behavior.

INTRODUCTION

Since their introduction, classical controls tools have been popular for analysis and design of single–input, single–output (SISO) systems. These methods may be viewed as specialized versions of more general tools that are applicable to multi–input, multi–output (MIMO) systems. Although modern "state–space" control methods (relying on dynamic models of internal structure) are generally promoted as the predominant tools for multivariable system analysis, the classical control extensions offer several advantages, including requiring only an input–output map and providing direct insight into stability, performance, and robustness of MIMO systems. The understanding generated by these graphically–based methods for the analysis and design of MIMO systems is a prime motivator of this research.

An early graphical method for investigating the stability of linear, time–invariant (LTI) SISO systems was developed by Nyquist (1932) and is based on a polar plot of the loop transmission transfer function. The MIMO analog of the Nyquist diagram is the multivariable Nyquist diagram which is used in conjunction with the corresponding multivariable Nyquist criterion (Rosenbrock, 1974; Lehtomaki, *et al.*, 1981; Friedland, 1986). This criterion is complicated in the MIMO case because it is expressed in terms of the determinant of the return difference transfer function matrix ($[\mathbf{I} + \mathbf{G}(s)]$ where $\mathbf{G}(s)$ is the plant transfer function matrix, rather than just $1 + g(s)$ for the SISO case where $g(s)$ is the plant transfer function). Despite the complication, significant research has supported the MIMO Nyquist extension for assessment of multivariable system stability and robustness (MacFarlane and Postlethwaite, 1977).

The Bode plots (Bode, 1940) recast the information of the Nyquist diagram, with frequency extracted as an explicit parameter. The MIMO analog or extension of the classical Bode magnitude plot is the singular value Bode–type plot that shows maximum and minimum singular values of transfer function matrices as a function of frequency (Doyle and Stein, 1981). This generalized magnitude *vs.* frequency plot is useful for analysis, providing performance insight in terms of command following, disturbance rejection, and sensor noise sensitivity, as well as for design, in terms of frequency shaping (Doyle and Stein, 1981; Safanov, *et al.*, 1981; Athans, 1982; Maciejowski, 1989).

Although promoted as an SISO tool, Evans root locus method (Evans, 1954) is also extendable to MIMO systems, since it depicts the trajectories of closed–loop

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complex plane. However, the generalization to the multivariable root loci has not made as significant an impact as the MIMO versions of the classical frequency-domain tools. In similarity to the multivariable Nyquist diagram and Bode plot, the MIMO root locus plot does not, in general, follow the straight–forward sketching rules applicable to SISO systems. However, it does provide insight into stability and performance of the closed–loop system. Part of the complication of the MIMO root locus relates to the fact that "multivariable root loci live on a Riemann surface ... as compared with the single–input, single–output case where the root loci lie on a simple complex plane (a trivial, *i.e.*, one sheeted, Riemann surface)" (Postlethwaite and MacFarlane, 1979). As a result, multivariable root loci tend to have strange looking patterns when drawn in a single complex plane. The possibility of loci being multi–valued functions of gain can make the MIMO root locus plot somewhat confusing.

To aid the controls engineer in extracting more information from the multivariable Evans root locus plot, we propose a set of "gain plots" that provide a direct and unique window into the stability, performance, and robustness of LTI MIMO systems. A conceptual framework motivating the gain plots and a discussion of their applicability to SISO systems has been presented previously (Kurfess and Nagurka, 1991a).

MULTIVARIABLE EIGENVALUE DESCRIPTION

Review of Basic MIMO Concepts

A LTI MIMO system can be represented in the standard state–space form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \tag{2}$$

where state vector $\mathbf{x}$ is length n, control input vector $\mathbf{u}$ is length m, and output vector $\mathbf{y}$ is length, m. Matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are the system matrix, the control influence matrix, the output matrix, and the feed–forward matrix, respectively, with appropriate dimensions. The input–output dynamics are governed by a square transfer function matrix, $\mathbf{G}(s)$,

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \tag{3}$$

The system is embedded in the closed–loop configuration, shown in Figure 1, where the controller is a static compensator, $k\mathbf{I}$, implying that each input channel is scaled by the same constant gain k. (Note that the plant transfer function matrix and any dynamic compensation may be combined in the transfer function matrix $\mathbf{G}(s)$.) The control law is given by

$$\mathbf{u}(t) = k\mathbf{I}\,\mathbf{e}(t) \tag{4}$$

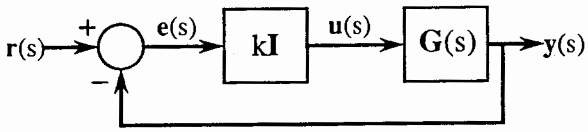

Figure 1. MIMO Closed-Loop Negative Feedback Configuration

where

$$\mathbf{e}(t) = \mathbf{r}(t) - \mathbf{y}(t) \tag{5}$$

is the error and $\mathbf{r}(t)$ is the reference (command) signal vector of length m that $\mathbf{y}(t)$ must track. The closed–loop transfer function matrix is

$$\mathbf{G}_{CL}(s) = [\,\mathbf{I} + k\,\mathbf{G}(s)]^{-1} k\,\mathbf{G}(s) \tag{6}$$

In the MIMO root locus plot, the migration of the eigenvalues of $\mathbf{G}_{CL}(s)$ in the complex plane is graphed as scalar k varies in the range $0 \le k < \infty$. The eigenvalues of the closed–loop system, $s = \lambda_i\,(i=1,2,...,n)$, are the roots of $\phi_{CL}(s)$, the closed–loop characteristic polynomial,

$$\phi_{CL}(s) = \phi_{OL}(s)\,\det[\mathbf{I} + k\mathbf{G}(s)] \tag{7}$$

where $\phi_{OL}(s)$ is the open–loop characteristic polynomial,

$$\phi_{OL}(s) = \det[s\mathbf{I} - \mathbf{A}] \tag{8}$$

The roots, or solutions of equation (8), are the open–loop poles. By equating the determinant in equation (7) to zero, the MIMO generalization of the SISO characteristic equation ($1 + kg(s) = 0$) is obtained. The presence of the determinant is the major challenge in generalizing the SISO root locus sketching rules to MIMO systems and complicates the root locus plot. For example, the root locus branches "move" between several copies (Riemann sheets) in the s–plane that are connected at singularity points known as branch points (Yagle, 1981; Athans, 1982).

Although it is not generally possible to sketch MIMO root loci by inspection, the closed–loop system eigenvalues may be computed numerically from equations (1) – (5) as

$$\lambda_i = \mathrm{eig}\left[\mathbf{A} - \mathbf{B}(\mathbf{I} + k\mathbf{D})^{-1}k\mathbf{C}\right]\ , \qquad i = 1, 2, \cdots, n \tag{9}$$

In the examples, the loci of the eigenvalues are calculated from equation (9) as k is monotonically increased from zero.

High Gain Behavior

As the gain increases from zero to infinity, the closed–loop eigenvalues trace out "root loci" in the complex plane. At zero gain, the poles of the closed–loop system are the open–loop eigenvalues. At infinite gain some of the eigenvalues approach finite transmission zeros, defined to be those values of s that satisfy the generalized eigenvalue problem

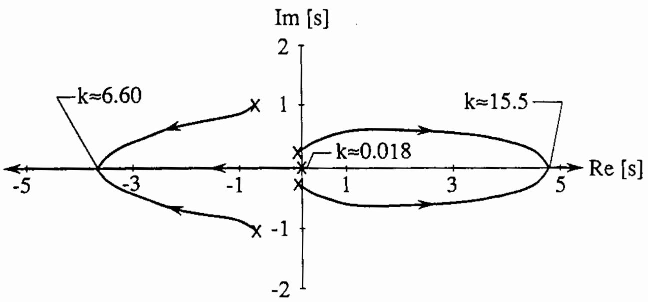

Figure 2. Root Locus Plot of Example 1

the system becomes unstable, but fails to indicate the gain at which instability occurs. Closer inspection of the eigenvalues indicates that the closed–loop system is never stable for positive gains.

The gain plots for this system, shown in Figure 3a,b, reveal this information about the closed–loop system instability. For example, they show that as the gain increases the eigenvalue at the origin initially migrates along the positive real axis (*i.e.*, $\angle s = 0°$), indicating instability, until it reaches a maximum value of s≈0.010 at a gain k≈0.018. As the gain increases, this real eigenvalue reverses direction, crosses the imaginary axis at a gain k≈0.043, and continues to move along the negative real axis (*i.e.*, $\angle s = 180°$). However, at k≈0.043 one pair of complex conjugate eigenvalues has already moved into the right half plane (crossing the imaginary axis at a slightly lower gain). The angle gain plot of Figure 3b shows this behavior clearly. In summary, the gain plots provide an unambiguous means by which stability may be determined.

The gain plots highlight several other important features. For example, they show the gains corresponding to the complex conjugate eigenvalue pairs breaking into the real axis and then proceeding to ±∞. Complex conjugate eigenvalues are shown as symmetric lines about either the 180° or 0° line with equal magnitudes. Purely real eigenvalues possess equal angles (180° or 0°) but distinct magnitudes. This behavior is demonstrated in Figure 3a,b, from which the gains at the breakpoints may be determined by inspection.

The rates at which the eigenvalues increase towards a magnitude of infinity is seen in the magnitude gain plot of Figure 3a and in expanded form in Figure 4. The single eigenvalue that begins at the origin proceeds towards infinity along the negative real axis at a rate proportional to k (the high gain magnitude gain plot slope is unity). This slope is characteristic of a first order Butterworth pattern. The two complex conjugate eigenvalue pairs proceed toward infinity at a rate proportional to $k^{1/2}$ (shown as a high gain magnitude gain plot slope of 1/2), indicative of a second order Butterworth pattern (Kurfess and Nagurka, 1991b).

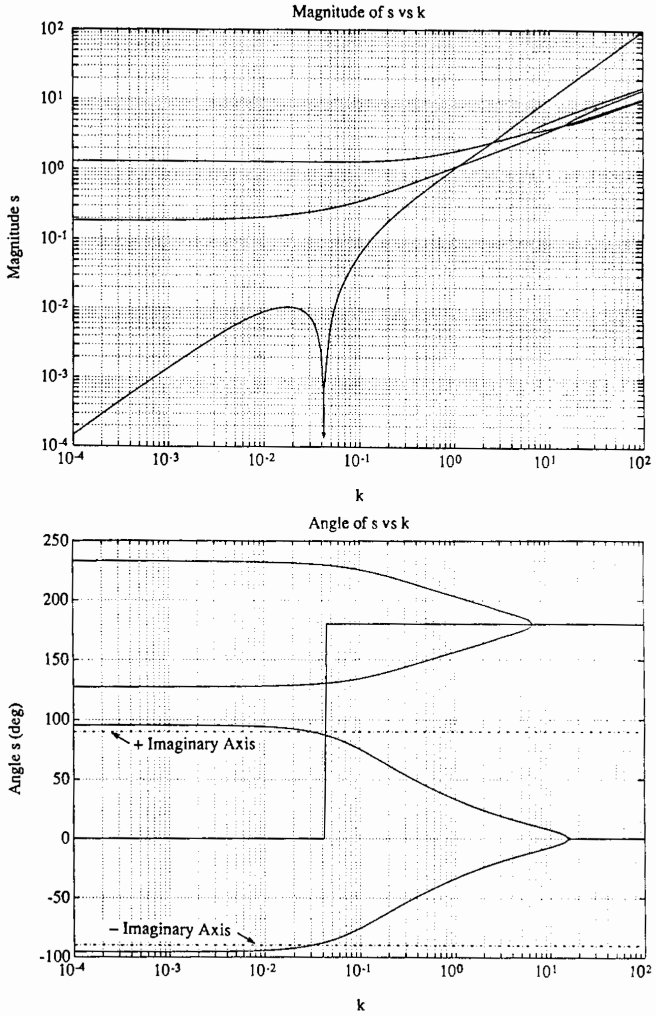

Figure 3. (a) Magnitude and (b) Angle Gain Plots of Example 1

From Figure 4, the two complex conjugate eigenvalue pairs at high gains have slope values of 1/2. As k→∞, this group of four parallel lines separates into two co–linear sets. An interesting fact is that the two identical lines are comprised of an eigenvalue magnitude from each of the original complex conjugate pairs. It is as if the complex conjugate eigenvalues have *swapped partners*. This phenomenon is not apparent from the MIMO root locus; however, it must occur due to the location of the centers of gravity for the two second order Butterworth patterns. In fact, each set of co–linear trajectories represents a Butterworth configuration.

Example 2: Higher Order System with Feedforward Term

This example, from (Kouvaritakis and Edmunds, 1979), demonstrates the power of the gain plot geometry in exposing multivariable system behavior. It represents a three input, three output, seventh order

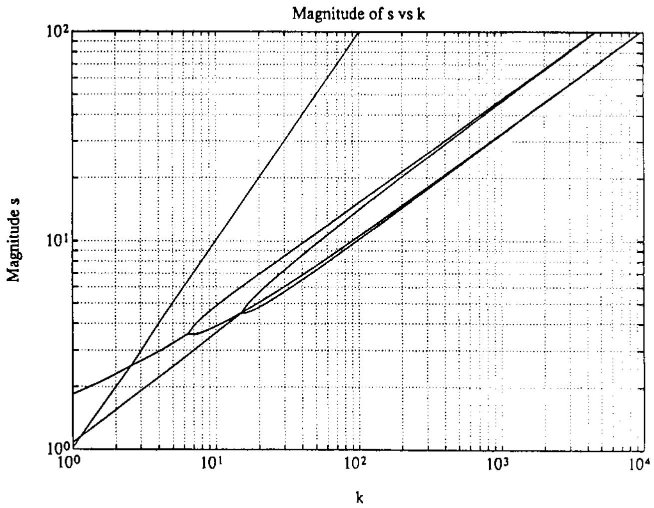

Figure 4. Expanded Magnitude Gain Plot of Example 1

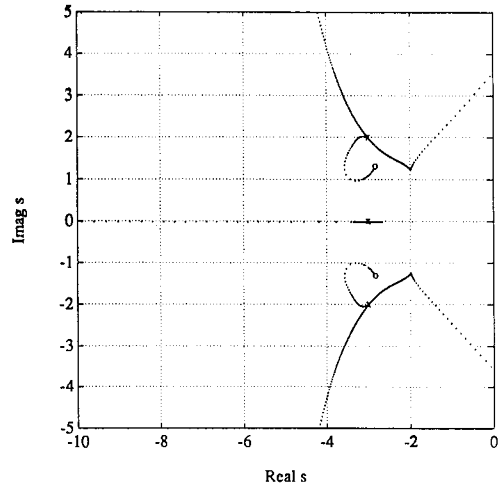

Figure 5. Root Locus Plot of Example 2

system with three transmission zeros. The system is given by the state space representation of equations (1) and (2) where

$$\mathbf{A} = \frac{1}{16}\begin{bmatrix} -32 & -80 & 16 & 0 & 0 & 0 & 0 \\ 16 & -64 & -16 & 0 & 0 & 0 & 0 \\ 0 & 0 & -48 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -32 & -80 & 0 & 0 \\ 0 & 0 & 0 & 16 & -64 & 0 & 0 \\ 1653 & 0 & 0 & 3424 & 0 & -32 & -80 \\ 76 & 0 & 0 & 928 & 0 & 16 & -64 \end{bmatrix} \qquad (13)$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -2 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (14)$$

$$\mathbf{C} = \frac{1}{16}\begin{bmatrix} 8 & -16 & 0 & 0 & 0 & 0 & 0 \\ -8 & -8 & 16 & -15 & -37 & 8 & 0 \\ 0 & 8 & 16 & -68 & 36 & 0 & 8 \end{bmatrix} \qquad (15)$$

$$\mathbf{D}' = \begin{bmatrix} -8 & -2 & 10 \\ -4 & -1 & 5 \\ -8 & -2 & 10 \end{bmatrix} \qquad (16)$$

The system is somewhat unusual due to the presence of the feedforward term (*i.e.*, the **D** matrix is non–zero).

The root locus plot for this system is shown in Figure 5, and the multivariable gain plots for the system are depicted in Figures 6a,b.

From the root locus and gain plots, it is clear that there are three sets of complex conjugate open–loop eigenvalues at $s = -3 \pm 2j$, and a single real open–loop eigenvalue at $s = -3$. There is also a set of complex conjugate multivariable transmission zeros $s \approx -2.84 \pm 1.31j$ and a real zero at $s \approx -124$. As the control configuration of Figure 1 is employed, the real eigenvalue moves first in the negative direction and then in the positive direction along the real axis. Inspection of the gain plots shows that the real eigenvalue reaches a maximum value of approximately –2.5 at a gain of approximately 0.1. The eigenvalue then branches to a different Riemann sheet and traverses along the real axis towards the real transmission zero. By inspection of the gain plots, pole/zero cancellation for the real zero occurs at $k \approx 10^4$.

A set of complex conjugate eigenvalues moves towards the transmission zeros as the gain is increased. The remaining two sets of eigenvalue pairs travel towards infinity in two separate second order Butterworth configurations. By inspection of the gain plots, pole/zero cancellation for the complex conjugate zeros occurs at a gain $k \approx 10^0$.

To further highlight the enriched perspective offered by the gain plots, a MIMO root locus plot for higher gain values is shown in Figure 7. (Because of the logarithmic scales used in the gain plots, expanded high gain plots are not necessary.) From Figure 7 the Butterworth patterns may not be clearly visible, yet from Figures 6a,b two distinct patterns arise. From the magnitude gain plot, the two separate configurations may be separated into two second order patterns having slopes of 1/2 (Kurfess and Nagurka, 1991b).

Further insight into the different patterns is available from the information of the gain plots. Although there are two sets of complex conjugate

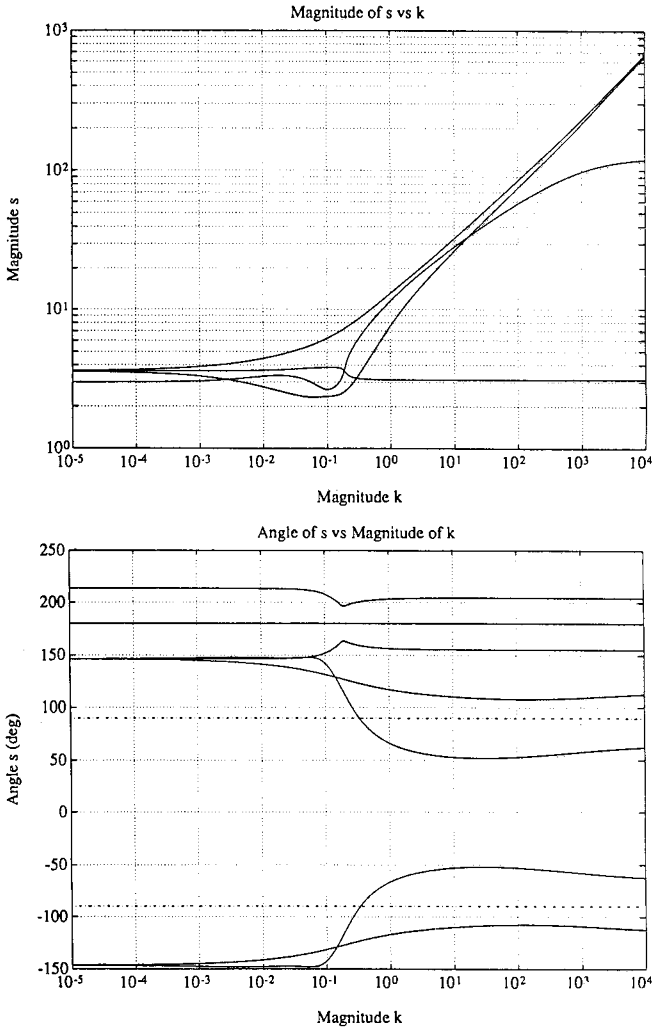

Figure 6. (a) Magnitude and (b) Angle Gain Plots of Example 2

eigenvalues, the Butterworth patterns are formed from one eigenvalue of each complex set. This is demonstrated in both the magnitude and angle gain plots. The angles of the complex conjugate eigenvalues are approximately ±115° and ±65°. Thus, each member complex pair is approximately 180° in angular distance from its matching Butterworth partner in the other complex pair. From simple geometric relationships, the centers of gravity (sometimes referred to as pivots) from the two second order Butterworth patterns may be computed to be approximately 19.2±11.9j (Wang, *et al.*, 1991).

CONCLUSIONS

In typical MIMO root locus plots trajectories may be camouflaged as some branches may overlap. Gain plots are promoted as a means to "untangle" MIMO

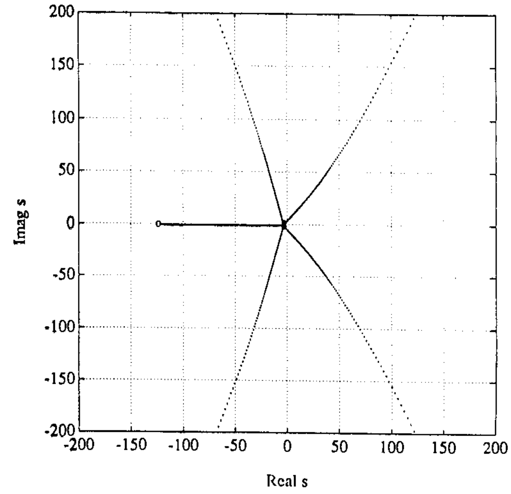

Figure 7. Expanded Range Root Locus Plot of Example 2

eigenvalue trajectories. The major enhancement is the visualization of eigenvalue trajectories as an explicit function of gain (where the compensation has been assumed to be the same static gain applied to all control channels). The representation provides a unique description of the eigenvalues and their trajectories as a parameter, such as gain, is varied.

Research efforts, currently underway, may shed additional light on gain plots for multivariable systems. In addition, work by MacFarlane and Postlethwaite (1977, 1979) and Hung and MacFarlane (1982) on relating characteristic frequency plots to gain domain geometry promises closer connections between gain plot methods and singular value frequency methods.

In conclusion, gain plots enrich the multivariable root locus plot in much the same way that singular value frequency plots are an alternate and extended presentation of the multivariable Nyquist diagram. Their use in conjunction with the multivariable root locus provides a new geometric perspective on multivariable systems that can result in clearer understandings of such systems in both the research and teaching realms of control engineering.

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