# SOLVING LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS BY CHEBYSHEV-BASED STATE PARAMETERIZATION

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## Abstract

*A Chebyshev-based state representation method is developed for solving optimal control problems involving unconstrained linear time-invariant dynamic systems with quadratic performance indices. In this method, each state variable is represented by the superposition of a finite-term shifted Chebyshev series and a third order polynomial. In contrast to solving a two-point boundary-value problem, here the necessary condition of optimality is a system of linear algebraic equations which can be solved by a method such as Gaussian elimination. The results of simulation studies demonstrate that the proposed method offers computational advantages relative to a previous Chebyshev method and to a standard state transition method.*

## 1. Introduction

Optimal control strategies can be applied to achieve the optimal control and the associated optimal state trajectories of linear, lumped parameter models of dynamic systems. These optimal trajectories are often determined from the necessary condition of optimality, which can be posed as a two-point boundary-value problem (TPBVP) using variational methods. By applying the Hamilton-Jacobi approach the TPBVP can be converted to a terminal value problem involving a matrix differential Riccati equation. Although this approach casts the optimal solution in closed-loop form making it a preferred approach for physical implementation, it can be computationally intensive and sometimes difficult to apply in solving high order systems.

For time-invariant systems, a more efficient solution method for optimal trajectory planning is the open-loop transition matrix approach (Speyer, 1986). This approach, which requires the calculation of a matrix exponential, converts the TPBVP into an initial value problem. The transition matrix approach can also encounter a problem, that of numerical instability, in determining the optimal control of high order systems (Yen and Nagurka, 1990). This problem has been attributed principally to the error associated with the computation of large dimension state transition matrices. An accurate and computationally streamlined approach for calculation of state transition matrices of high order systems remains a research challenge (Moler and Loan, 1978).

To circumvent these numerical difficulties, which often complicate and in some cases prevent the solution of the TPBVP, trajectory parameterization methods have been proposed. In general, these approaches approximate the control, state, and/or co-state trajectories by finite-term orthogonal functions whose unknown coefficient values are sought giving a near optimal (or sub-optimal) solution. For example, approaches employing functions such as Walsh (Chen and Hsiao, 1975), block-pulse (Hsu and Cheng, 1981), Laguerre (Shih, Kung and Chao, 1986), Chebyshev (Paraskevopoules, 1983; Vlassenbroeck and Van Dooren, 1988), and Fourier (Chung, 1987) have been suggested. Many of these approaches, in similarity to the state transition matrix approach, employ algorithms that convert the TPBVP into an initial value problem. The initial value problem is then reduced to a static optimization problem represented by algebraic equations by approximating the state and co-state vectors by truncated orthogonal functions. Although the truncation results in errors that can be minimized by including more terms, the transition matrix (needed to convert the TPBVP to an initial value problem) must still be evaluated which, as mentioned above, can cause instability problems in high order systems.

This research is part of a broader effort toward the development of computational tools for solving optimal control problems via state parameterization. An advantage of state parameterization is that boundary condition requirements on the state variables can be satisfied directly. A second advantage is that the state equations can be treated as algebraic equations in determining the corresponding control trajectory. This assumes that no constraints on the control structure prevent an arbitrary representation of the state trajectory from being achieved.

Earlier work on parameterization of the state vector via Fourier-type series (Yen and Nagurka, 1988) has shown that the necessary condition of optimality for an unconstrained linear quadratic (LQ) problem can be formulated as a system of linear algebraic equations. To ensure an arbitrary representation of the state trajectory and hence overcome the potential difficulty of trajectory inadmissibility (due to constraints preventing an arbitrary state trajectory), artificial control variables were proposed. These physically non-existent variables are driven small by being heavily penalized in the performance index. Simulation results indicated that the approach is accurate, computationally efficient, and robust relative to standard methods.

Studies of parameterization methods for prediction of the optimal control of linear time-invariant systems have demonstrated advantages of expansions in terms of Chebyshev functions in comparison to Walsh, block-pulse, Hermite, Laguerre, and Legendre functions (Paraskevopoulos, 1983,1985). Chebyshev functions can nearly uniformly approximate a broad class of functions, making them computationally attractive (Vlassenbroeck and Van Dooren, 1988).

This paper explores a method based on a finite-term Chebyshev representation of the state trajectory. By applying this method, the necessary condition of optimality is derived as a system of linear algebraic equations from which the unknown state parameter vector can be solved. In comparison to a previous Chebyshev-based method and to a state transition matrix method, the proposed approach is computationally efficient, especially for high-order systems.

## 2. Methodology

### 2.1 Problem Statement

Given a linear dynamic system with the state-space model

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad , \quad x(0) = x_0 \tag{1}$$



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with known initial condition, the design goal is to find the control $\mathbf{u}(t)$ and the corresponding state $\mathbf{x}(t)$ in the time interval [0,T] that minimizes the quadratic performance index L,

$$L = L_1 + L_2 \tag{2}$$

where

$$L_1 = \mathbf{x}^T(T)\mathbf{H}\mathbf{x}(T) + \mathbf{h}^T\mathbf{x}(T) \tag{3}$$

$$L_2 = \int_0^T [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t) + \mathbf{x}^T(t)\mathbf{S}(t)\mathbf{u}(t) + \mathbf{q}^T(t)\mathbf{x}(t) + \mathbf{r}^T(t)\mathbf{u}(t)]dt \tag{4}$$

It is assumed that $\mathbf{x}$ is an N×1 state vector, $\mathbf{u}$ is an M×1 control vector, $\mathbf{A}$ is an N×N system matrix, $\mathbf{B}$ is an N×M control influence matrix, $\mathbf{H}$ and $\mathbf{Q}$ are real N×N symmetric and positive-semidefinite weighting matrices, $\mathbf{R}$ is an M×M symmetric and positive definite weighting matrix, $\mathbf{S}$ is an N×M weighting matrix, and $\mathbf{h}$, $\mathbf{q}$ and $\mathbf{r}$ are N×1 weighting vectors. In addition, it is assumed that the lengths of the state and control vectors are the same (*i.e.*, M=N) and $\mathbf{B}$ is invertible.

### 2.2 Chebyshev Polynomials

Chebyshev polynomials are defined for the interval $\xi \in [-1,1]$ and have the following analytical form:

$$\varphi_k(\xi) = \cos(k \cos^{-1}\xi) \quad , \quad k = 0, 1, 2, \ldots \tag{5}$$

or

$$\varphi_k(\xi) = \sum_{i=0}^{[k/2]} (-1)^i \frac{k!}{(2i)!(k-2i)!}(1-\xi)^i \xi^{k-2i} \quad , \quad k = 0, 1, 2, \ldots \tag{6}$$

where the notation [k/2] means the greatest integer smaller than k/2.

The domain of the Chebyshev polynomials can be transformed to values between 0 and T by letting $\xi = 2t/T-1$ giving the shifted Chebyshev polynomial $\psi_k(t)$, where

$$\psi_k(t) = \varphi_k(\xi) = \varphi_k(2t/T-1) \tag{7}$$

From Equations (6) and (7), the first few shifted Chebyshev polynomials are

$$\psi_0(t) = 1 \quad , \quad \psi_1(t) = 2\tau-1 \quad , \quad \psi_2(t) = 8\tau^2-8\tau+1 \tag{8a-c}$$

where nondimensional time $\tau = t/T$. From Equations (5) and (7), the initial and final values of the shifted Chebyshev polynomial and its first time derivative can be obtained as

$$\psi_k(0) = (-1)^k \quad , \quad \dot{\psi}_k(0) = (-1)^{k+1}(2k^2/T)$$
$$\psi_k(T) = 1 \quad , \quad \dot{\psi}_k(T) = 2k^2/T) \tag{9a-d}$$

### 2.3 Chebyshev-Based State Parameterization

2.3.1 State Parameterization In this Chebyshev-based state parameterization approach, each of the N state variables $x_n(t)$, n=1,2,...,N, is approximated by the sum of a third-order auxiliary polynomial and a (K−4) term shifted Chebyshev series.

$$x_n(t) = b_{n0} + b_{n1}\tau + b_{n2}\tau^2 + b_{n3}\tau^3 + \sum_{k=4}^{K-1} a_{nk}\psi_k(t) \tag{10}$$

A motivation for this representation is that the boundary values of the state variables can be decoupled from the unknown state parameters enabling the state initial condition to be satisfied directly. The derivative of $x_n(t)$ is then

$$\dot{x}_n(t) = \frac{1}{T}\left(b_{n1} + 2b_{n2}\tau + 3b_{n3}\tau^2\right) + \sum_{k=4}^{K-1} a_{nk}\dot{\psi}_k(t) \tag{11}$$

The constants b's can be determined by substituting the initial and final values of time (0 and T) into Equations (10) and (11), using Equations (9a-d), and manipulating.

$$b_{n0} = x_{n0}-(-1)^k \sum_{k=4}^{K-1} a_{nk} \quad , \quad b_{n1} = T\dot{x}_{n0}+(-1)^k \sum_{k=4}^{K-1} 2k^2 a_{nk}$$

$$b_{n2} = -3x_{n0}-2T\dot{x}_{n0}+3x_{nT}-T\dot{x}_{nT}+\sum_{k=4}^{K-1} \left[(-3+2k^2)+(-1)^k(3-4k^2)\right]a_{nk}$$

$$b_{n3} = 2x_{n0}+T\dot{x}_{n0}-2x_{nT}+T\dot{x}_{nT}+\sum_{k=4}^{K-1} \left[(2-2k^2)-(-1)^k(2-2k^2)\right]a_{nk} \tag{12a-d}$$

where $x_{n0}$, $\dot{x}_{n0}$, $x_{nT}$, and $\dot{x}_{nT}$ are the values of the state variable $x_n$ and its derivative $\dot{x}_n$ at the boundaries of the time segment [0,T], *i.e.*, $x_{n0} = x_n(0)$, $\dot{x}_{n0} = \dot{x}_n(0)$, $x_{nT} = x_n(T)$, and $\dot{x}_{nT} = \dot{x}_n(T)$. By substituting Equations (12a-d) into (10), $x_n(t)$ can be rearranged as

$$x_n(t) = \sum_{k=1}^{K} c_k(t) y_{nk} \tag{13}$$

where

$$c_1 = 1-3\tau^2+2\tau^3 \quad , \quad c_2 = T(\tau-2\tau^2+\tau^3) \tag{14a,b}$$

$$c_3 = 3\tau^2-2\tau^3 \quad , \quad c_4 = T(-\tau^2+\tau^3) \tag{15a,b}$$

$$c_k = (-1)^k - 2(-1)^k(k-1)^2\tau + [2k^2-4k-1+(-1)^k(4k^2-8k+1)]\tau^2 + 2k(2k)[1+(-1)]\tau^3 + \psi_{k-1}(t) \quad (k=5,6,\ldots,K) \tag{16}$$

and where

$$y_{n1} = x_{n0} \quad , \quad y_{n2} = \dot{x}_{n0} \quad , \quad y_{n3} = x_{nT} \quad , \quad y_{n4} = \dot{x}_{nT} \tag{17a-d}$$

$$y_{nk} = a_{n(k-1)} \quad (k=5,6,\ldots,K) \tag{18}$$

Equation (13) can be written more compactly as

$$x_n(t) = \mathbf{c}^T(t)\mathbf{y}_n \tag{19}$$

where

$$\mathbf{c}^T(t) = \left[c_1(t) \;\; c_2(t) \;\; \cdots \;\; c_K(t)\right] \tag{20}$$

$$\mathbf{y}_n = \left[y_{n1} \;\; y_{n2} \;\; \cdots \;\; y_{nK}\right]^T \tag{21}$$

Vector $\mathbf{y}_n$ is the state parameter vector for the n-th state variable. (Except for $y_{n1}$, the state parameter vector $\mathbf{y}_n$ contains unknown elements.)

The N state variables of the state vector can be written in terms of a full state parameter vector $\mathbf{y}$, *i.e.*,

$$\mathbf{x}(t) = \mathbf{C}(t)\mathbf{y} \tag{22}$$

where

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{c}^T(t) & & & 0 \\ & \mathbf{c}^T(t) & & \\ & & \ddots & \\ 0 & & & \mathbf{c}^T(t) \end{bmatrix}_{N\times NK} \tag{23}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} [y_{11} & y_{12} & \cdots & y_{1K}]^T \\ [y_{21} & y_{22} & \cdots & y_{2K}]^T \\ & \vdots & & \\ [y_{N1} & y_{N2} & \cdots & y_{NK}]^T \end{bmatrix}_{NK\times 1} \tag{24}$$

Similarly,

$$\dot{\mathbf{x}}(t) = \mathbf{D}(t)\mathbf{y} \tag{25}$$

where

$$\mathbf{D}(t) = \dot{\mathbf{C}}(t) = \begin{bmatrix} \mathbf{d}^T(t) & & 0 \\ & \mathbf{d}^T(t) & \\ & & \ddots \\ 0 & & \mathbf{d}^T(t) \end{bmatrix}_{N\times NK} \tag{26}$$

$$\mathbf{d}^T(t) = [\dot{c}_1(t)\ \dot{c}_2(t)\ \cdots\ \dot{c}_K(t)] \tag{27}$$

The control $\mathbf{u}(t)$ can also be expressed as a function of $\mathbf{y}$. From Equation (1),

$$\mathbf{u}(t) = \mathbf{B}^{-1}(t)\dot{\mathbf{x}}(t) - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{x}(t) \tag{28}$$

From Equations (22) and (25),

$$\mathbf{u}(t) = [\mathbf{B}^{-1}(t)\mathbf{D}(t) - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{C}(t)]\mathbf{y} \tag{29}$$

Thus, the state vector, state rate vector, and control vector can be represented as functions of the state parameter vector.

2.3.2 Conversion Process The unconstrained LQ problem can be converted to a quadratic programming (QP) problem via the proposed Chebyshev-based state parameterization. The first step is to rewrite the performance index as a function of the state parameter vector $\mathbf{y}$. The terminal state vector $\mathbf{x}(T)$ can be expressed from Equation (22) and then substituted into Equation (3) giving the terminal cost $L_1$ as a quadratic in terms of $\mathbf{y}$.

$$L_1 = \mathbf{y}^T\mathbf{C}(T)^T\mathbf{H}\mathbf{C}(T)\mathbf{y} + \mathbf{h}^T\mathbf{C}(T)\mathbf{y} \tag{30}$$

Similarly, by substituting Equation (28) into the integrand of Equation (4),

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{x}^T\mathbf{S}\mathbf{u} + \mathbf{q}^T\mathbf{x} + \mathbf{r}^T\mathbf{u} = \mathbf{x}^T\mathbf{F}_1\mathbf{x} + \dot{\mathbf{x}}^T\mathbf{F}_2\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{F}_3\mathbf{x} + \mathbf{x}^T\mathbf{f}_1 + \dot{\mathbf{x}}^T\mathbf{f}_2 \tag{31}$$

where $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ are N×N matrices and $\mathbf{f}_1$ and $\mathbf{f}_2$ are N×1 vectors given by

$$\mathbf{F}_1 = \mathbf{Q} + \mathbf{G}^T\mathbf{R}\mathbf{G} + \mathbf{S}\mathbf{G} \quad , \quad \mathbf{F}_2 = \mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1} \tag{32a-b}$$

$$\mathbf{F}_3 = 2\mathbf{B}^{-T}\mathbf{R}\mathbf{G} + \mathbf{B}^{-T}\mathbf{S} \tag{33}$$

$$\mathbf{f}_1 = \mathbf{q} + \mathbf{G}^T\mathbf{r} \quad , \quad \mathbf{f}_2 = \mathbf{B}^{-T}\mathbf{r} \tag{34a-b}$$

where $\mathbf{G} = -\mathbf{B}^{-1}\mathbf{A}$ and superscript $-T$ denotes inverse transpose. By substituting Equations (22) and (25) into (31), the integrand of Equation (4) can be expressed as a quadratic in terms of $\mathbf{y}$, *i.e.*,

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{x}^T\mathbf{S}\mathbf{u} + \mathbf{q}^T\mathbf{x} + \mathbf{r}^T\mathbf{u} = \mathbf{y}^T\mathbf{P}\mathbf{y} + \mathbf{y}^T\mathbf{p} \tag{35}$$

where

$$\mathbf{P} = \mathbf{F}_1 \otimes \mathbf{c}\mathbf{c}^T + \mathbf{F}_2 \otimes \mathbf{d}\mathbf{d}^T + \mathbf{f}_3 \otimes \mathbf{d}\mathbf{c}^T \tag{36}$$

$$\mathbf{p} = \mathbf{f}_1 \otimes \mathbf{c} + \mathbf{f}_2 \otimes \mathbf{d} \tag{37}$$

Here, $\mathbf{P}$ is an NK×NK matrix, $\mathbf{p}$ is an NK×1 matrix, and $\otimes$ is the Kronecker product sign (Brewer, 1978). From Equation (35), the integral part of the performance index can be expressed as

$$L_2 = \int_0^T (\mathbf{y}^T\mathbf{P}\mathbf{y} + \mathbf{y}^T\mathbf{p})dt = \mathbf{y}^T\mathbf{P}^*\mathbf{y} + \mathbf{y}^T\mathbf{p}^* \tag{38}$$

where

$$\mathbf{P}^* = \int_0^T \mathbf{P}\,dt \quad , \quad \mathbf{p}^* = \int_0^T \mathbf{p}\,dt \tag{39a-b}$$

Substituting Equations (30) and (38) into (2) gives the performance index L as a quadratic in terms of $\mathbf{y}$, *i.e.*,

$$L = \mathbf{y}^T\Omega\mathbf{y} + \mathbf{y}^T\omega \tag{40}$$

where

$$\Omega = \mathbf{C}(T)^T\mathbf{H}\mathbf{C}(T) + \mathbf{P}^* \quad , \quad \omega = \mathbf{C}(T)^T\mathbf{h} + \mathbf{p}^* \tag{41a-b}$$

For time-invariant problems, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are constants and can be removed from the integrals, enabling the remaining integral parts of $\mathbf{P}^*$ and $\mathbf{p}^*$ to be evaluated analytically. That is, Equation (39a-b) can be rewritten as

$$\mathbf{P}^* = \mathbf{F}_1 \otimes \left[\int_0^T (\mathbf{c}\mathbf{c}^T)dt\right] + \mathbf{F}_2 \otimes \left[\int_0^T (\mathbf{d}\mathbf{d}^T)dt\right] + \mathbf{F}_3 \otimes \left[\int_0^T (\mathbf{d}\mathbf{c}^T)dt\right]$$

$$\mathbf{p}^* = \mathbf{f}_1 \otimes \left[\int_0^T \mathbf{c}\,dt\right] + \mathbf{f}_2 \otimes \left[\int_0^T \mathbf{d}\,dt\right] \tag{42a-b}$$

The solutions of the integral parts of $\mathbf{P}^*$ and $\mathbf{p}^*$ (*i.e.*, the terms in the brackets) have been determined in closed-form and have been summarized as recurrence formulas.

2.3.3 Solution Procedure The problem can be viewed as an optimization problem involving the search for the unknown coefficients of the state parameter vector $\mathbf{y}$ that minimize Equation (40) subject to the equality constraints of Equation (17a). To isolate the free variables, a new state parameter vector $\mathbf{z}$ is introduced as

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}_{NK\times 1} \tag{43}$$

where

$$\mathbf{z}_1^T = [\mathbf{a}^T\ \dot{\mathbf{x}}_0^T\ \dot{\mathbf{x}}_T^T\ \mathbf{x}_T^T] \quad , \quad \mathbf{z}_2 = \mathbf{x}_0 \tag{44a-b}$$

with

$$\mathbf{x}_0^T = [x_{10}\ x_{20} \cdots x_{N0}] \quad , \quad \dot{\mathbf{x}}_0^T = [\dot{x}_{10}\ \dot{x}_{20} \cdots \dot{x}_{N0}]$$

$$\mathbf{x}_T^T = [x_{1T}\ x_{2T} \cdots x_{NT}] \quad , \quad \dot{\mathbf{x}}_T^T = [\dot{x}_{1T}\ \dot{x}_{2T} \cdots \dot{x}_{NT}] \tag{45a-d}$$

$$\mathbf{a}^T = [a_{14}\ a_{15} \cdots a_{1(K-1)}\ a_{24}\ a_{25} \cdots a_{2(K-1)} \cdots a_{N4} \cdots a_{N(K-1)}]$$

$$= [y_{15}\ y_{16} \cdots y_{1K}\ y_{25}\ y_{26} \cdots y_{2K} \cdots y_{N5} \cdots y_{NK}] \tag{46}$$

Vector $\mathbf{z}_2$ contains the known initial values of the state vector and vector $\mathbf{z}_1$ is the remaining subset of the parameter vector $\mathbf{y}$. The two vectors $\mathbf{z}$ and $\mathbf{y}$ are related via a linear transformation:

$$\mathbf{y} = \Phi\mathbf{z} \tag{47}$$

where $\Phi$ is an NK×NK matrix with elements 1 and 0.

Given Equation (47), L in Equation (40) can be rewritten as a function of $\mathbf{z}$.

$$L = z^T \Omega^* z + z^T \omega^* \quad (48)$$

where

$$\Omega^* = \Phi^T \Omega \Phi \quad , \quad \omega^* = \Phi^T \omega \quad (49a\text{-}b)$$

By expanding Equation (48), the performance index can be expressed as

$$L = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix} \begin{bmatrix} \Omega_{11}^* & \Omega_{12}^* \\ \Omega_{21}^* & \Omega_{22}^* \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} z_1^T & z_2^T \end{bmatrix} \begin{bmatrix} \omega_1^* \\ \omega_2^* \end{bmatrix} \quad (50)$$

or, equivalently,

$$L = z_1^T \Omega_{11}^* z_1 + z_1^T \left(\Omega_{12}^* + \Omega_{21}^{*T}\right) z_2 + z_2^T \Omega_{22}^* z_2 + z_1^T \omega_1^* + z_2^T \omega_2^* \quad (51)$$

The necessary condition of optimality can be obtained by differentiating L with respect to the unknown state parameter vector $z_1$. This leads to

$$\left(\Omega_{11}^* + \Omega_{11}^{*T}\right) z_1 = -\left(\Omega_{12}^* + \Omega_{21}^{*T}\right) z_2 - \omega_1^* \quad (52)$$

which represents a system of linear algebraic equations from which the unknown vector $z_1$ can be solved.

### 2.4 Chebyshev-Based Approach for General Linear Systems

The approach presented above is applicable to systems with square and invertible control influence matrices. For general linear systems, **B** is an N×M matrix where N is greater than M. To apply the Chebyshev-based approach, Equation (1) is modified to

$$\dot{x}(t) = A(t)x(t) + B'(t)u'(t) \quad (53)$$

where

$$B'(t) = B'_{N\times N} = \begin{bmatrix} \begin{matrix} I_{(N-M)\times(N-M)} \\ 0_{M\times(N-M)} \end{matrix} & B_{N\times M} \end{bmatrix} \quad (54)$$

$$u'(t) = u'_{N\times 1} = \begin{bmatrix} \hat{u}_{(N-M)\times 1} \\ u_{M\times 1} \end{bmatrix} \quad (55)$$

where $\hat{u}$ is an artificial (*i.e.*, fictitious) control vector.

It can be guaranteed that B′ is invertible if the last M rows of B are nonsingular. However, if the last M rows are singular, the first (N-M) columns of B′ in Equation (54) can always be modified to make it invertible. In order to predict the optimal solution, the performance index is modified to

$$L' = L_1 + L_2' \quad (56)$$

where $L_1$ is given by Equation (3) and where

$$L_2' = \int_0^T [x^T(t)Q(t)x(t) + u'^T(t)R'(t)u'(t) + x^T(t)S'(t)u'(t) + q^T(t)x(t) + r'^T(t)u'(t)]dt \quad (57)$$

with

$$R'(t) = R'_{N\times N} = \begin{bmatrix} \rho I_{(N-M)\times(N-M)} & 0_{(N-M)\times M} \\ 0_{M\times(N-M)} & R_{M\times M} \end{bmatrix}$$

$$S'(t) = S'_{N\times N} = \begin{bmatrix} \begin{matrix} \rho I_{(N-M)\times(N-M)} \\ 0_{M\times(N-M)} \end{matrix} & S_{N\times M} \end{bmatrix} \quad (58a\text{-}c)$$

$$r'(t) = r'_{N\times N} = \begin{bmatrix} \rho & \cdots & \rho & r^T \end{bmatrix}^T$$

where ρ is a weighting constant chosen to be a large positive number. If **S**=**0**, **q**=**0** and **r**=**0**, then Equation (56) simplifies to

$$L' = L_1 + \int_0^T [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt + \rho \int_0^T \left[\hat{u}^T(t)\hat{u}(t)\right]dt \quad (59)$$

By penalizing the artificial control vector, the magnitude and influence of the artificial control variables can be made small and the solution of the modified optimal control problem can approximate the solution of the original LQ problem.

## 3. Simulation Study

This example considers an N input N-th order linear time-invariant dynamic system expressed in canonical form.

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x^T(0) = [1 \;\; 2 \;\cdots\; N] \quad (60)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \; u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & -2 & \cdots & (-1)^{N+1}N \end{bmatrix}, \; B = I_{N\times N} \quad (61a\text{-}d)$$

The problem is to find **u**(t) that minimizes the performance index

$$L = x^T(1)H\,x(1) + \int_0^1 \left(x^TQx + u^TRu\right)dt \quad (62)$$

where

$$H = 10\, I_{N\times N} \quad , \quad Q = R = I_{N\times N} \quad (63a\text{-}b)$$

One of the most efficient methods for solving this unconstrained LQ problem is the transition matrix approach (details can be found in (Speyer, 1986)). The approach converts an optimal control problem into a linear TPBVP. By evaluating the transition matrix of this boundary value problem, the problem can be converted into an initial value problem. In this study, the transition matrices were computed numerically using the algorithm presented in (Franklin and Powell, 1980).

**Table 1. Comparison of Simulation Results for Example [1]**

| N | Transition Matrix | | Other Chebyshev | | Chebyshev-based | |
|---|---|---|---|---|---|---|
| | Perf Index | Time (sec) | %Error [2] | %Time [3] | %Error [2] | %Time [3] |
| 2 | 5.3591 | 0.50 | 1.29e-03 | 144.0 | 3.21e-05 | 156.0 |
| 4 | 44.249 | 2.42 | 2.56e-03 | 171.9 | 7.67e-04 | 67.8 |
| 6 | 153.75 | 7.06 | 3.67e-02 | 187.0 | 5.23e-03 | 48.4 |
| 8 | 373.02 | 15.86 | 1.56e-01 | 193.8 | 1.84e-02 | 40.7 |
| 10 | 741.61 | 29.04 | 1.76e-01 | 202.8 | 4.41e-02 | 38.2 |
| 12 | 1299.3 | 50.44 | 1.74e-01 | 204.0 | 8.32e-02 | 35.1 |
| 14 | 2086.3 | 81.46 | 1.61e-01 | 198.3 | 1.34e-01 | 33.2 |
| 16 | 3142.8 | 124.54 | 1.48e-01 | 197.0 | 1.94e-01 | 30.8 |
| 18 | 4509.0 | 174.24 | 1.39e-01 | 199.6 | 2.61e-01 | 30.4 |
| 20 | 6225.4 | 247.50 | 1.36e-01 | 191.8 | 3.31e-01 | 28.7 |

[1] Six-term series for Other Chebyshev approach. For Chebyshev-based approach, four terms are used in polynomial and two terms in Chebyshev series.

[2] Magnitude of percent relative error of Chebyshev performance index with respect to Transition Matrix performance index

[3] Percent of execution time of Chebyshev approach relative to execution time of Transition Matrix approach

An alternate approach is a Chebyshev approach adapted from (Paraskevopoulos, 1983). It also converts an optimal control problem into an initial value problem. Then, the state and costate vectors in the linear homogeneous differential equations are expanded in Chebyshev series with unknown coefficients. By integrating the differential equations and introducing a "Chebyshev operational matrix", the unknown coefficients of the Chebyshev series may be determined. The state and control vectors may then be obtained. (In this study, the linear algebraic equations were solved by an LU-decomposition routine.) For comparison, this approach - henceforth referred to as the "other Chebyshev" approach - was implemented.

In addition to the transition matrix approach and the other Chebyshev approach, the proposed Chebyshev-based approach of Section 2.3 was implemented to solve this problem. A Gauss-Jordan elimination routine was used to solve the linear algebraic equations representing the conditions of optimality in Equation (52). A two-term shifted Chebyshev series in conjunction with a third-order polynomial was assumed.

Efforts were made to optimize the speeds of the computer codes, all of which were written in "C" and executed on a SUN-3/60 workstation. Simulation results for N=2,4,...,20 are summarized in Table 1. For the transition matrix and Chebyshev-based approaches, the execution time includes the time to evaluate *(i)* the system response (state and control vectors) at 100 equally-spaced points and *(ii)* the performance index. For the other Chebyshev approach, the execution time includes only the time to evaluate the system response. (The table reports execution time for the transition matrix approach in seconds, and percent execution time relative to the time of the transition matrix approach for the other and Chebyshev methods.)

The results show that the Chebyshev-based approach is the computationally most attractive approach with the relative error of the performance index less than one percent. In comparison to the transition matrix approach, the Chebyshev-based approach is increasingly more efficient for N>2. For N=20, the Chebyshev-based results suggest greater than 70 percent savings in execution time. For N=2, the Chebyshev-based method is less efficient than the transition matrix approach since the time to evaluate the integrals in Equations (42a-b), a fixed time for any order system, is a significant fraction of the overall computation cost. For high order systems the principal computational cost is due to the solution of the linear algebraic equation (52), which is less intensive than the solution via the transition matrix method.

The other Chebyshev approach is computationally more costly than the transition matrix approach. In this approach the relative error of the performance index does not grow significantly when the order of the system increases. The execution time is approximately twice the time of the transition matrix approach.

The time histories of the state and control variables for the case N=2 are plotted in Figures 1a and 1b, respectively. The response curves from the transition matrix and the Chebyshev-based approaches drawn in these figures overlap for the scale shown. Hence, the Chebyshev-based solutions achieve convergence on the trajectories of the state and control variables as well as on the value of the performance index.

## 4. Conclusions

This paper presents a state parameterization method based on a finite-term Chebyshev representation for predicting the (near) optimal state and control trajectories of unconstrained linear time-invariant dynamic systems with quadratic performance indices. In the proposed method, the time history of each state variable is represented by the superposition of a shifted Chebyshev series and a third order polynomial. The necessary condition of optimality gives a system of linear algebraic equations from which the unknown state parameters can be solved. The results of simulation studies demonstrate computational advantages of the proposed Chebyshev method relative to a previous Chebyshev method and a standard state transition matrix approach.

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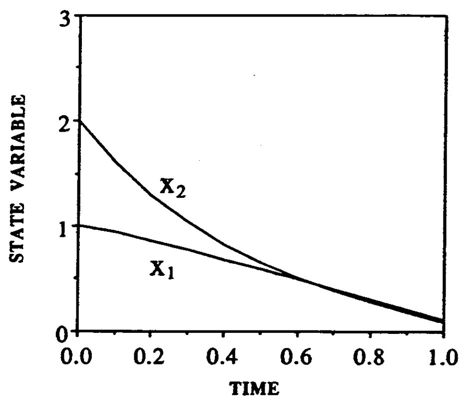


**Figure 1a** State Variable History for Example

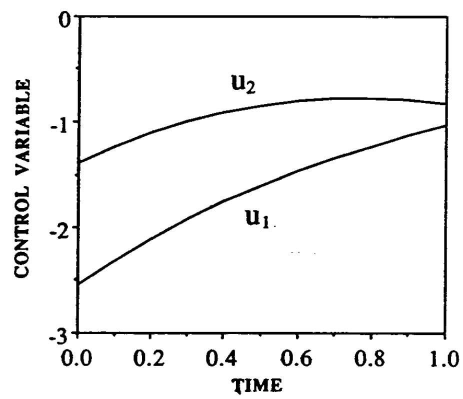


**Figure 1b** Control Variable History for Example

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