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Optimal Trajectory Planning via Chebyshev-Based State Parameterization

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Two state parameterization methods based on finite-term Chebyshev representations are developed to determine the optimal state and control trajectories of unconstrained linear time-invariant dynamic systems with quadratic performance indices. In one method, each state variable of a dynamic system is approximated by a shifted Chebyshev series. In the second method, each state variable is represented by the superposition of a shifted Chebyshev series and a third order polynomial. In both approaches, the necessary and sufficient condition of optimality is derived as a system of linear algebraic equations. The results of simulation studies demonstrate that the Chebyshev-plus-polynomial method offers computational advantages relative to the direct Chebyshev method, a previous Chebyshev method and a state-transition approach.

1. Introduction

The optimal control, and corresponding state, trajectories of linear, lumped parameter models of dynamic systems are often determined from the necessary condition of optimality. Using variational methods, this optimality condition can be represented as a two-point boundary-value problem (TPBVP). One of the most well-known solution approaches is the Hamilton-Jacobi approach which converts the TPBVP to a terminal value problem involving a matrix differential Riccati equation. Although the Hamilton-Jacobi approach casts the optimal solution in closed-loop form making it a preferred approach for physical implementation, it is computationally intensive and sometimes difficult to employ in solving high order systems.

For time-invariant systems, a more efficient solution method for optimal trajectory planning is the open-loop transition matrix approach [1], which converts the TPBVP into an initial value problem. The transition matrix approach can also encounter a problem, that of numerical instability, in determining the optimal control of high order systems [2].

To circumvent these numerical difficulties, and in the interest of seeking alternative solution strategies, trajectory parameterization methods have been investigated. In general, these approaches approximate the control, state, and/or co-state trajectories by finite-term orthogonal functions whose unknown coefficient values are sought giving a near optimal (or sub-optimal) solution. For example, approaches employing functions such as Walsh [3], block-pulse [4], Laguerre [5], Chebyshev [6-8], and Fourier [9-10] representations have been suggested. Many of these approaches employ algorithms that convert the TPBVP into an initial value problem whose state and co-state vectors are then approximated by truncated orthogonal series. This technique reduces the initial value problem to a static optimization problem represented by algebraic equations.

Earlier work involving parameterization of the state vector via Fourier-type series [10] has shown that the necessary condition of optimality for an unconstrained linear quadratic (LQ) problem can be formulated as a system of linear algebraic equations. To ensure an arbitrary representation of the state trajectory and hence overcome the potential difficulty of trajectory inadmissibility (due to constraints on the control structure preventing an arbitrary state trajectory), artificial control variables were proposed. These physically non-existent variables are driven small by being heavily penalized in the performance index. Simulation results indicated that the approach is accurate, computationally efficient, and robust relative to standard methods.

Studies of parameterization methods for optimal control of linear time-invariant systems have demonstrated advantages of expansions in terms of Chebyshev functions in comparison to other functions [6]. Chebyshev functions can nearly uniformly approximate a broad class of functions, making them computationally attractive [8].

This paper explores two methods based on finite-term Chebyshev representations of the state trajectory. In one method, the direct Chebyshev method, each state variable of a dynamic system is approximated by a shifted Chebyshev series. The unconstrained LQ problem is then converted to an equality constrained quadratic programming (QP) problem that minimizes the performance index and satisfies state initial conditions via Lagrange multipliers. In the second method, a modified and improved Chebyshev method, each state variable is represented by the superposition of a shifted Chebyshev series and a third order polynomial. In both methods the necessary condition of optimality gives a system of linear algebraic equations from which the unknown state parameter vector can be solved.

2. Methodology

2.1 Problem Statement

The behavior of a linear dynamic system is governed by the state-space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \quad (1)$$

with known initial condition $\mathbf{x}(0)=\mathbf{x}_0$ where $\mathbf{x}$ is an N×1 state vector, $\mathbf{u}$ is an M×1 control vector, $\mathbf{A}$ is an N×N system matrix, and $\mathbf{B}$ is an N×M control influence matrix. For now, it is assumed that M=N and $\mathbf{B}$ is invertible. These assumptions will be relaxed later.

The problem is to plan the trajectories of the control $\mathbf{u}(t)$ and the corresponding state $\mathbf{x}(t)$ in the time interval [0,T] that minimizes the quadratic performance index L,

$$L = L_1 + L_2 \quad (2)$$

where

$$L_1 = \mathbf{x}^T(T)\mathbf{H}\mathbf{x}(T) + \mathbf{h}^T\mathbf{x}(T) \quad (3)$$

$$L_2 = \int_0^T [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t)+\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)+\mathbf{x}^T(t)\mathbf{S}(t)\mathbf{u}(t)+\mathbf{q}^T(t)\mathbf{x}(t)+\mathbf{r}^T(t)\mathbf{u}(t)]dt \quad (4)$$

without violating the linear system constraints:

$$\mathbf{E}_1(t)\mathbf{x}(t) + \mathbf{E}_2(t)\mathbf{u}(t) \le \mathbf{e}(t) \quad (5)$$

It is assumed that **H**, **Q**, **R** and **S** are real N×N symmetric matrices with **H** and **Q** being positive-semidefinite and **R** being positive definite, **h**, **q** and **r** are N×1 vectors, **e** is a J×1 vector, $\mathbf{E}_1$ is a J×N matrix, and $\mathbf{E}_2$ is a J×M matrix.

2.2 Chebyshev Polynomials

Chebyshev polynomials are defined for the interval $\xi \in [-1,1]$ and have the following analytical form:

$$\varphi_k(\xi) = \cos(k \cos^{-1}\xi) \quad , \quad k = 0, 1, 2, \ldots \quad (6)$$

or

$$\varphi_k(\xi) = \sum_{i=0}^{[k/2]} (-1)^i \frac{k!}{(2i)!(k-2i)!}(1-\xi)^i \xi^{k-2i} \quad , \quad k = 0, 1, 2, \ldots \quad (7)$$

where the notation [k/2] means the greatest integer smaller than k/2. From Equation (7), the first few Chebyshev polynomials are

$$\varphi_0(\xi) = 1 \ , \ \varphi_1(\xi) = \xi \ , \ \varphi_2(\xi) = 2\xi^2-1$$
$$\varphi_3(\xi) = 4\xi^3-3\xi \ , \ \varphi_4(\xi) = 8\xi^4-8\xi^2+1 \ , \ \varphi_5(\xi) = 16\xi^5-20\xi^3+5\xi \quad (8a\text{-}f)$$

The Chebyshev polynomials have several properties, such as satisfying *(i)* the recurrence relations

$$\varphi_{k+1}(\xi) - 2\xi\varphi_k(\xi) + \varphi_{k-1}(\xi) = 0 \quad , \quad k = 1, 2, \ldots$$
$$\left(1-\xi^2\right)\dot{\varphi}_k(\xi) = -k\xi\varphi_k(\xi) + k\varphi_{k-1}(\xi) \quad , \quad k = 1, 2, \ldots \quad (9a\text{-}b)$$

where the dot indicates differentiation with respect to time, *(ii)* the initial, final and midpoint values

$$\varphi_k(1) = 1 \quad , \quad \varphi_k(-1) = (-1)^k$$
$$\varphi_{2k}(0) = (-1)^k \ , \ \varphi_{2k+1}(0) = 0 \quad (10a\text{-}d)$$

and *(iii)* the product relations

$$\varphi_i(\xi)\varphi_j(\xi) = \frac{1}{2}[\varphi_{i+j}(\xi)+\varphi_{i-j}(\xi)] \quad , \quad i \ge j$$
$$\varphi_k^2(\xi) = \frac{1}{2}[1+\varphi_{2k}(\xi)] \quad (11a\text{-}b)$$

The domain of the Chebyshev polynomials can be transformed to values between 0 and T by letting

$$\xi = 1-2\frac{t}{T} \quad (12)$$

giving the shifted Chebyshev polynomial $\psi_k(t)$ expressed as

$$\psi_k(t) = \varphi_k(\xi) = \varphi_k\left(1-2\frac{t}{T}\right) \quad (13)$$

From Equation (13) the first few shifted Chebyshev polynomials are

$$\psi_0(t) = 1 \ , \ \psi_1(t) = -2\tau+1 \ , \ \psi_2(t) = 8\tau^2-8\tau+1$$
$$\psi_3(t) = -32\tau^3+48\tau^2-18\tau+1 \ , \ \psi_4(t) = 128\tau^4-256\tau^3+160\tau^2-32\tau+1 \quad (14a\text{-}e)$$

where nondimensional time $\tau = t / T$. The initial and final values of the shifted Chebyshev polynomial and its first time derivative can be obtained as

$$\psi_k(0) = 1 \quad , \quad \dot{\psi}_k(0) = -2k^2/ T \quad (15a\text{-}b)$$

$$\psi_k(T) = (-1)^k \quad , \quad \dot{\psi}_k(T) = (-1)^k(2k^2/ T) \quad (16a\text{-}b)$$

2.3 Direct Chebyshev-Based State Parameterization

2.3.1 State Parameterization A direct approach for Chebyshev-based state parameterization is to approximate each of the N state variables $x_n(t)$ by a K term shifted Chebyshev series.

$$x_n(t) = \sum_{k=1}^{K} c_k(t)\, y_{nk} \quad (17)$$

for n=1,2,...,N where

$$c_k(t) = \psi_{k-1}(t) \quad (18)$$

In Equation (17) y_{nk} is the k-th unknown coefficient for the n-th state variable. Equation (17) can be written as

$$x_n(t) = \mathbf{c}^T(t)\mathbf{y}_n \quad (19)$$

where

$$\mathbf{c}^T(t) = [c_1(t) \ c_2(t) \ \cdots \ c_K(t)] \quad (20)$$

$$\mathbf{y}_n = [y_{n1} \ y_{n2} \ \cdots \ y_{nK}]^T \quad (21)$$

The state vector containing the N state variables can be written in terms of a full state parameter vector **y**, *i.e.*,

$$\mathbf{x}(t) = \mathbf{C}(t)\mathbf{y} \quad (22)$$

where

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{c}^T(t) & & 0 \\ & \mathbf{c}^T(t) & \\ & & \ddots \\ 0 & & \mathbf{c}^T(t) \end{bmatrix}_{N\times NK} \quad (23)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} [y_{11} & y_{12} & \cdots & y_{1K}]^T \\ [y_{21} & y_{22} & \cdots & y_{2K}]^T \\ & \vdots & & \\ [y_{N1} & y_{N2} & \cdots & y_{NK}]^T \end{bmatrix}_{NK\times 1} \quad (24)$$

Similarly,

$$\dot{\mathbf{x}}(t) = \mathbf{D}(t)\mathbf{y} \quad (25)$$

where

$$\mathbf{D}(t) = \dot{\mathbf{C}}(t) = \begin{bmatrix} \mathbf{d}^T(t) & & 0 \\ & \mathbf{d}^T(t) & \\ & & \ddots \\ 0 & & \mathbf{d}^T(t) \end{bmatrix}_{N\times NK} \quad (26)$$

$$\mathbf{d}^T(t) = [\dot{c}_1(t) \ \dot{c}_2(t) \ \cdots \ \dot{c}_K(t)] \quad (27)$$

The control vector **u**(t) can also be expressed as a function of **y**. From Equation (1),

$$\mathbf{u}(t) = \mathbf{B}^{-1}(t)\dot{\mathbf{x}}(t)-\mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{x}(t) \quad (28)$$

From Equations (22) and (25),

$$\mathbf{u}(t) = [\mathbf{B}^{-1}(t)\mathbf{D}(t)-\mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{C}(t)]\mathbf{y} \quad (29)$$

Thus, using the direct Chebyshev-based state parameterization approach the state, state rate, and control vectors can be represented as functions of the state parameter vector.

2.3.2 Conversion Process The first step in converting this unconstrained LQ problem to a quadratic programming (QP) problem via direct Chebyshev-based state parameterization is to rewrite the performance index as a function of the state parameter vector **y**. From Equation (22), the terminal state vector **x**(T) can be expressed as

$$x(T) = C(T)y \quad (30)$$

By substituting Equation (30) into (3), the cost L_1 is

$$L_1 = y^T C^T(T) H C(T) y + h^T C(T) y \quad (31)$$

Similarly, by substituting Equation (28) into the integrand of Equation (4),

$$x^T Qx+u^T Ru+x^T Su+q^T x+r^T u = x^T F_1 x+\dot{x}^T F_2 \dot{x}+\dot{x}^T F_3 x+x^T f_1+\dot{x}^T f_2 \quad (32)$$

where F_1, F_2, and F_3 are N×N matrices and f_1 and f_2 are N×1 vectors given by

$$F_1 = Q+G^T RG+SG \quad , \quad F_2 = B^{-T} RB^{-1} \quad , \quad F_3 = 2B^{-T} RG+B^{-T} S \quad (33a\text{-}c)$$

$$f_1 = q+G^T r \quad , \quad f_2 = B^{-T} r \quad (34a\text{-}b)$$

where

$$G = -B^{-1} A \quad (35)$$

and superscript $-T$ denotes inverse transpose. By substituting Equations (22) and (25) into (32), the integrand of Equation (4) can be expressed as a function of parameter vector **y**, *i.e.*,

$$x^T Qx+u^T Ru+x^T Su+q^T x+r^T u = y^T Py+y^T p \quad (36)$$

where

$$P = F_1 \otimes cc^T + F_2 \otimes dd^T + f_3 \otimes dc^T \quad , \quad p = f_1 \otimes c + f_2 \otimes d \quad (37a\text{-}b)$$

In Equations (37a-b), **P** is an NK×NK matrix, **p** is an NK×1 matrix, and ⊗ is a Kronecker product sign, *e.g.*, if V is an n×n matrix,

$$V \otimes w = \begin{bmatrix} V_{11}w & \cdots & V_{1n}w \\ V_{21}w & & \vdots \\ \vdots & & \vdots \\ V_{n1}w & \cdots & V_{nn}w \end{bmatrix} \quad (38)$$

From Equation (36), the integral part of the performance index can be expressed as

$$L_2 = \int_0^T (y^T P y + y^T p) dt = y^T P^* y + y^T p^* \quad (39)$$

where

$$P^* = \int_0^T P \, dt \quad , \quad p^* = \int_0^T p \, dt \quad (40a\text{-}b)$$

Substituting Equations (31) and (39) into (2) gives the performance index L as a quadratic function of parameter vector **y**, *i.e.*,

$$L = y^T \Omega y + y^T \omega \quad (41)$$

where

$$\Omega = C^T(T) H C(T) + P^* \quad , \quad \omega = C^T(T) h + p^* \quad (42a\text{-}b)$$

For time-invariant problems, F_1, F_2, F_3, f_1 and f_2 are constants and can be removed from the integrals, enabling the remaining integral parts of P^* and p^* to be evaluated analytically. That is, Equations (40a-b) can be rewritten as

$$P^* = F_1 \otimes \left[\int_0^T (c c^T) dt\right] + F_2 \otimes \left[\int_0^T (d d^T) dt\right] + F_3 \otimes \left[\int_0^T (d c^T) dt\right]$$

$$p^* = f_1 \otimes \left[\int_0^T c \, dt\right] + f_2 \otimes \left[\int_0^T d \, dt\right] \quad (43a\text{-}b)$$

The terms in the brackets have been determined in closed-form.

The initial conditions of the state variables can be expressed as

$$x_0 = C_0 y \quad (44)$$

where

$$x_0 = x(0) \quad , \quad C_0 = C(0) \quad (45a\text{-}b)$$

Hence, the problem is to minimize Equation (41) such that Equation (44) is satisfied.

2.3.3 Solution Procedure The above equality constrained QP problem is converted to an unconstrained problem by appending the constraints to the performance index via a Lagrange multiplier vector λ:

$$L(y,\lambda) = y^T \Omega y + y^T \omega + \lambda^T [C_0 y - x_0] \quad (46)$$

The necessary conditions of optimality are given by

$$\nabla_y L(y,\lambda) = (\Omega + \Omega^T) y + \omega + C_0^T \lambda = 0 \quad (47)$$

$$\nabla_\lambda L(y,\lambda) = C_0 y - x_0 = 0 \quad (48)$$

representing a system of linear algebraic equations in terms of the elements of **y** and λ. Equations (47) and (48) can be written as

$$\begin{bmatrix} \Omega + \Omega^T & C_0^T \\ C_0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \lambda \end{bmatrix} = \begin{bmatrix} -\omega \\ x_0 \end{bmatrix} \quad (49)$$

from which the state parameter vector **y** can be solved.

2.4 Improved Chebyshev-Based State Parameterization

An improved Chebyshev-based state parameterization is presented in this section. Here, each of the N state variables $x_n(t)$ is approximated by the sum of a third-order auxiliary polynomial and a (K-4) term shifted Chebyshev series. The motivation for including the auxiliary polynomial is based on results for Fourier series [11] that show that convergence on [0,T] can be guaranteed for **x**, $\dot{x}$, and $\ddot{x}$ and that the speed of convergence can be improved. By analogy, an auxiliary polynomial was employed here to extend the "differentiability" of the representation and improve the speed of convergence in comparison to a Chebyshev series (*e.g.*, as used in the previous direct method).

2.4.1 State Parameterization In contrast to the direct approach of Section 2.3.1, the **c** and y_n vectors of Equation (19) are redefined. First, consider a K term shifted Chebyshev series, *i.e.*, for n=1,2,...,N

$$x_n(t) = \sum_{k=0}^{K-1} a_{nk} \psi_k(t) \quad (50)$$

The state variable x_n and its derivative $\dot{x}_n$ at the boundaries of the time segment [0,T] are

$$x_{n0} = x_n(0) \quad , \quad \dot{x}_{n0} = \dot{x}_n(0) \quad , \quad x_{nT} = x_n(T) \quad , \quad \dot{x}_{nT} = \dot{x}_n(T) \quad (51a\text{-}d)$$

By substituting Equations (14a-d) into (50), $x_n(t)$ and its time derivative can be arranged as

$$x_n(t) = b_{n0} + b_{n1}\tau + b_{n2}\tau^2 + b_{n3}\tau^3 + \sum_{k=4}^{K-1} a_{nk} \psi_k(t)$$

$$\dot{x}_n(t) = \frac{1}{T}(b_{n1} + 2b_{n2}\tau + 3b_{n3}\tau^2) + \sum_{k=4}^{K-1} a_{nk} \dot{\psi}_k(t) \quad (52a\text{-}b)$$

where $\tau = t/T$ and the b's are new constants. They can be determined by substituting the initial and final values of time (0 and T) into Equations (52a-b), using Equations (16a-d) and (51a-d), and performing some algebraic manipulations.

$$b_{n0} = x_{n0} - \sum_{k=4}^{K-1} a_{nk} \quad , \quad b_{n1} = T\dot{x}_{n0} + \sum_{k=4}^{K-1} 2k^2 a_{nk}$$

$$b_{n2} = -3x_{n0} - 2T\dot{x}_{n0} + 3x_{nT} - T\dot{x}_{nT} + \sum_{k=4}^{K-1} \left[(3-4k^2) + (-1)^k(-3+2k^2)\right] a_{nk} \qquad (53a\text{-}d)$$

$$b_{n3} = 2x_{n0} + T\dot{x}_{n0} - 2x_{nT} + T\dot{x}_{nT} + \sum_{k=4}^{K-1} \left[(-2+2k^2) + (-1)^k(2-2k^2)\right] a_{nk}$$

By substituting Equations (53a-d) into (52a), Equation (50) can be rearranged as

$$x_n(t) = \sum_{k=1}^{K} c_k(t) y_{nk} \qquad (54)$$

where

$$c_1 = 1 - 3\tau^2 + 2\tau^3 \ , \ c_2 = T(\tau - 2\tau^2 + \tau^3) \ , \ c_3 = 3\tau^2 - 2\tau^3 \ , \ c_4 = T(-\tau^2 + \tau^3) \qquad (55a\text{-}d)$$

$$c_k = -1 + \eta_{1k}\tau + \eta_{2k}\tau^2 + \eta_{3k}\tau^3 + \psi_{k-1}(t) \quad (k=5, 6, \cdots, K) \qquad (56)$$

with

$$\eta_{1k} = 2(k-1)^2$$
$$\eta_{2k} = -4k^2 + 8k - 1 - (-1)^k(2k^2 - 4k - 1) \qquad (57a\text{-}c)$$
$$\eta_{3k} = 2k(k-2)[1 + (-1)^k]$$

and where

$$y_{n1} = x_{n0} \ , \ y_{n2} = \dot{x}_{n0} \ , \ y_{n3} = x_{nT} \ , \ y_{n4} = \dot{x}_{nT} \ , \ y_{nk} = a_{n(k-1)} \quad (k=5, 6, \cdots, K) \qquad (58a\text{-}e)$$

Then Equation (54) can be written as Equation (19) with different definitions of y_{nk}'s and c_k's.

2.4.2 Conversion Process Using Equations (58a-e) and (24), the terminal state vector $\mathbf{x}(T)$ can be expressed as

$$\mathbf{x}(T) = \Theta \mathbf{y} \qquad (59)$$

where the elements of the N×NK matrix Θ are

$$\theta_{ij} = \begin{cases} 1 & j=(i-1)K+3 \quad i=1,2,\cdots,N \\ 0 & \text{otherwise} \end{cases} \qquad (60)$$

By substituting Equation (59) into (3), the cost L_1 is

$$L_1 = \mathbf{y}^T \Theta^T \mathbf{H} \Theta \mathbf{y} + \mathbf{h}^T \Theta \mathbf{y} \qquad (61)$$

The conversion process is similar to the direct approach except that Equations (30) and (31) are replaced by Equations (59) and (61), respectively. Because the c_k's in Equations (55a-d) and (56) are redefined, the integrals of $\mathbf{P}^*$ and $\mathbf{p}^*$ change. For time-invariant problems new closed-form expressions for the integral parts can be derived.

2.4.3 Solution Procedure This equality constrained QP problem is solved by converting it into an unconstrained QP problem. A new state parameter vector $\mathbf{z}$ is introduced as

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}_{NK\times 1} \qquad (62)$$

where

$$\mathbf{z}_1^T = \left[\mathbf{a}^T \ \dot{\mathbf{x}}_0^T \ \dot{\mathbf{x}}_T^T \ \mathbf{x}_T^T\right] \quad , \quad \mathbf{z}_2 = \mathbf{x}_0 \qquad (63a\text{-}b)$$

$$\mathbf{x}_0^T = [x_{10} \ x_{20} \ \ldots \ x_{N0}] \quad , \quad \dot{\mathbf{x}}_0^T = [\dot{x}_{10} \ \dot{x}_{20} \ \ldots \ \dot{x}_{N0}]$$

$$\mathbf{x}_T^T = [x_{1T} \ x_{2T} \ \ldots \ x_{NT}] \quad , \quad \dot{\mathbf{x}}_T^T = [\dot{x}_{1T} \ \dot{x}_{2T} \ \ldots \ \dot{x}_{NT}] \qquad (64a\text{-}e)$$

$$\mathbf{a}^T = [a_{14} \ a_{15} \ldots a_{1(K-1)} \ a_{24} \ a_{25} \ldots a_{2(K-1)} \ldots a_{N4} \ldots a_{N(K-1)}]$$

$$= [y_{15} \ y_{16} \ldots y_{1K} \ y_{25} \ y_{26} \ldots y_{2K} \ldots y_{N5} \ldots y_{NK}]$$

Vector $\mathbf{z}_2$ contains the known initial values of the state vector and vector $\mathbf{z}_1$ is the remaining subset of the parameter vector $\mathbf{y}$. The two vectors $\mathbf{z}$ and $\mathbf{y}$ are related via a linear transformation:

$$\mathbf{y} = \Phi \mathbf{z} \qquad (65)$$

where Φ is an NK×NK matrix with elements

$$\phi_{ij} = 1 \qquad i = (n-1)K + k \quad i = 1, 2 \ldots, N; \quad k = 1, 2 \ldots, K$$

$$j = \begin{cases} NK - N + n & k = 1 \\ NK - 4N + n & k = 2 \\ NK - 2N + n & k = 3 \\ NK - 3N + n & k = 4 \\ (n-1)(K-4) + (k-4) & k = 5, 6, \ldots, K \end{cases} \qquad (66)$$

$$\phi_{ij} = 0 \ . \ \text{otherwise}$$

The performance index L in Equation (41) can thus be rewritten as a function of $\mathbf{z}$

$$L = \mathbf{z}^T \Omega^* \mathbf{z} + \mathbf{z}^T \omega^* \qquad (67)$$

where

$$\Omega^* = \Phi^T \Omega \Phi \quad , \quad \omega^* = \Phi^T \omega \qquad (68a\text{-}b)$$

By expanding Equation (67), the performance index can be expressed as

$$L = \left[\mathbf{z}_1^T \ \mathbf{z}_2^T\right] \begin{bmatrix} \Omega_{11}^* & \Omega_{12}^* \\ \Omega_{21}^* & \Omega_{22}^* \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \left[\mathbf{z}_1^T \ \mathbf{z}_2^T\right] \begin{bmatrix} \omega_1^* \\ \omega_2^* \end{bmatrix} \qquad (69)$$

or equivalently

$$L = \mathbf{z}_1^T \Omega_{11}^* \mathbf{z}_1 + \mathbf{z}_1^T \left(\Omega_{12}^* + \Omega_{21}^{*T}\right) \mathbf{z}_2 + \mathbf{z}_2^T \Omega_{22}^* \mathbf{z}_2 + \mathbf{z}_1^T \omega_1^* + \mathbf{z}_2^T \omega_2^* \qquad (70)$$

For an unconstrained LQ problem, the necessary condition of optimality can be obtained by differentiating the performance index with respect to the unknown state parameter vector, $\mathbf{z}_1$. This leads to

$$\left(\Omega_{11}^* + \Omega_{11}^{*T}\right)\mathbf{z}_1 = -\left(\Omega_{12}^* + \Omega_{21}^{*T}\right)\mathbf{z}_2 - \omega_1^* \qquad (71)$$

which represents a system of linear algebraic equations from which $\mathbf{z}_1$ can be solved.

2.5 General Linear Systems

The approaches presented above are applicable to systems with square and invertible control influence matrices. To apply the Chebyshev-based approach to the more common case of general linear systems which have fewer control variables than state variables, the state-space model of Equation (1) is modified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}'(t)\mathbf{u}'(t) \qquad (72)$$

where

$$\mathbf{B}'(t) = \mathbf{B}'_{N\times N} = \begin{bmatrix} \mathbf{I}_{(N-M)\times(N-M)} & \\ & \mathbf{B}_{N\times M} \\ \mathbf{0}_{M\times(N-M)} & \end{bmatrix} \qquad (73)$$

$$\mathbf{u}'(t) = \mathbf{u}'_{N\times 1} = \begin{bmatrix} \hat{\mathbf{u}}_{(N-M)\times 1} \\ \mathbf{u}_{M\times 1} \end{bmatrix} \qquad (74)$$

where $\hat{\mathbf{u}}$ is an artificial control vector.

It can be guaranteed that $\mathbf{B}'$ is invertible if the last M rows of $\mathbf{B}$ are nonsingular. However, if the last M rows are singular, the first (N-M) columns of $\mathbf{B}'$ can always be modified to make it invertible. In order to predict the optimal solution, the performance index is modified to

$$L' = L + \rho \int_0^T \left[\hat{\mathbf{u}}^T(t)\,\hat{\mathbf{u}}(t)\right] dt \tag{75}$$

where L is the performance index of the original LQ problem and ρ is a weighting constant chosen to be a large positive number. By penalizing the artificial control vector, the magnitude and influence of the artificial control variables can be made small and the solution of the modified optimal control problem can approximate the solution of the original LQ problem.

3. Simulation Study

To study the effectiveness of the approaches, the solutions of unconstrained, time-invariant LQ problems have been obtained by both the direct and improved Chebyshev-based state parameterization approaches and compared with solutions from other numerical algorithms.

3.1 Example

This example considers an N input N-th order linear time-invariant dynamic expressed in canonical form.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad , \quad \mathbf{x}^T(0) = [1 \;\; 2 \;\cdots\; N] \tag{76}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & -2 & \cdots & (-1)^{N+1}N \end{bmatrix} , \quad \mathbf{B} = \mathbf{I}_{N\times N} \tag{77a-b}$$

The problem is to determine the control $\mathbf{u}$ that minimizes

$$L = \mathbf{x}^T(1)\mathbf{H}\,\mathbf{x}(1) + \int_0^1 (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\, dt \tag{78}$$

where

$$\mathbf{H} = 10\,\mathbf{I}_{N\times N} \quad , \quad \mathbf{Q} = \mathbf{R} = \mathbf{I}_{N\times N} \tag{79a-b}$$

One of the most efficient methods commonly used for solving this unconstrained LQ problem is the transition matrix approach [1]. The approach converts an optimal control problem into a linear TPBVP. By evaluating the transition matrix of this boundary value problem, the problem can be converted into an initial value problem which can readily be solved. In this study, the transition matrices were computed numerically using the algorithm presented in [12].

An alternate approach is a Chebyshev approach adapted from [6]. It also converts an optimal control problem into a linear initial value problem. Then, the state and co-state vectors in the linear homogeneous differential equations are expanded in Chebyshev series. By integrating the differential equations and introducing a "Chebyshev operational matrix", the unknown coefficients of the Chebyshev series may be determined which enables the near optimal state and control trajectories to be obtained. For comparison, this approach - henceforth referred to as the "other Chebyshev" approach - was implemented.

In addition to the transition matrix approach and the other Chebyshev approach, the direct and improved Chebyshev-based approaches (with six term series) were implemented to solve the example problem. These approaches are described in Sections 2.3 and 2.4. The Gauss-Jordan elimination routine was used to solve the linear algebraic equations representing the conditions of optimality in Equations (49) and (71).

Efforts were made to optimize the speeds of the computer codes, all of which were written in "C" and executed on a SUN-3/60 workstation. Simulation results for N=2,4,...,20 are summarized in Table 1. For the transition matrix, direct and improved Chebyshev-based approaches, the execution time is the time to evaluate *(i)* the system response (control vector) at 100 equally-spaced points and *(ii)* the performance index. For the other Chebyshev approach, the execution time is only the time to evaluate the system response. (The table reports execution time for the transition matrix approach in seconds, and percent execution time relative to the time of the transition matrix approach for the other, direct, and improved Chebyshev methods.)

The results show that the improved Chebyshev-based approach is the computationally most attractive approach with the relative error of the performance index less than one percent. In comparison to the transition matrix approach, the improved Chebyshev-based approach is increasingly more efficient for N>2. For N=20, the improved Chebyshev-based results suggest greater than 70 percent savings in execution time. For N=2, the improved Chebyshev-based method is less efficient than the

Table 1. Comparison of Simulation Results.*

*** Six-term series for Other, Direct and Improved Chebyshev approaches.**

N	Transition Matrix		Other Chebyshev		Direct Chebyshev		Improved Chebyshev	
	PI	Time	PI%error	%Time	PI%error	%Time	PI%error	%Time
2	5.3591	0.50	-1.29E-03	144.0	3.21E-05	120.0	3.21E-05	156.0
4	44.249	2.42	-2.56E-03	171.9	7.67E-04	81.0	7.67E-04	67.8
6	153.75	7.06	3.67E-02	187.0	5.23E-03	73.4	5.23E-03	48.4
8	373.02	15.86	1.56E-01	193.8	1.84E-02	69.4	1.84E-02	40.7
10	741.61	29.04	1.76E-01	202.8	4.41E-02	70.0	4.41E-02	38.2
12	1299.3	50.44	1.74E-01	204.0	8.32E-02	67.4	8.32E-02	35.1
14	2086.3	81.46	1.61E-01	198.3	1.34E-01	64.8	1.34E-01	33.2
16	3142.8	124.54	1.48E-01	197.0	1.94E-01	62.4	1.94E-01	30.8
18	4509.0	174.24	1.39E-01	199.6	2.61E-01	62.8	2.61E-01	30.4
20	6225.4	247.50	1.36E-01	191.8	3.31E-01	60.2	3.31E-01	28.7

transition matrix approach since the time to evaluate the integrals in Equations (43a-b), a fixed time for any order system, is a significant fraction of the overall computation cost. For high order systems the principal computational cost is due to the solution of the linear algebraic equation (71), which is less intensive than the solution via the transition matrix method.

The direct Chebyshev-based approach offers less time savings than the improved Chebyshev-based approach for high order systems, but is still much faster than the transition matrix approach. The direct Chebyshev-based approach is more efficient than the improved Chebyshev-based approach when N<4 since the integrals in Equations (43a-b) are easier to solve. Both the direct and improved approaches have the same values for the performance indices and control vectors because the terms of the series used to approximate the state variables are the same.

The other Chebyshev approach is computationally more costly than the transition matrix approach. The advantage of this approach is that the relative error of the performance index does not grow significantly when the system order increases. The time is approximately twice the time of the transition matrix approach.

The time histories of the state and control variables for N=2 are plotted in Figures 1a and 1b, respectively. The curves from the transition matrix and direct/improved Chebyshev-based approaches drawn in these figures overlap for the scale shown. Hence, the direct and improved Chebyshev-based solutions achieve convergence on the trajectories of the state and control variables and on the value of the performance index.

4. Conclusions

This paper presents two state parameterization methods based on finite-term Chebyshev representations of the state trajectory. Such representations are used for predicting the optimal state and control trajectories of unconstrained linear time-invariant dynamic systems with quadratic performance indices. In one method, the direct Chebyshev method, the time history of each state variable is approximated by a shifted Chebyshev series. The unconstrained LQ problem is then converted to an equality constrained QP problem that minimizes the performance index and satisfies the state initial conditions via Lagrange multipliers. In the second method, a modified and improved Chebyshev method, the time history of each state variable is represented by the superposition of a shifted Chebyshev series and a third order polynomial. The inclusion of the auxiliary polynomial improves the speed of convergence and the "differentiability" of the representation in comparison to a standard Chebyshev series. In both methods, the necessary condition of optimality gives a system of linear algebraic equations from which the unknown state parameters can be solved. The results of simulation studies demonstrate computational advantages of the improved Chebyshev method relative to the direct method, a previous Chebyshev method and a standard state transition matrix approach.

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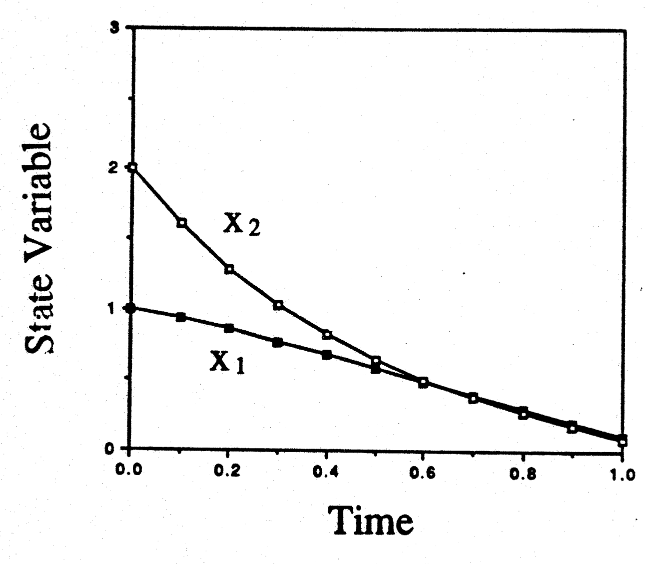

Figure 1a. State Variable Histories of Example (N=2).

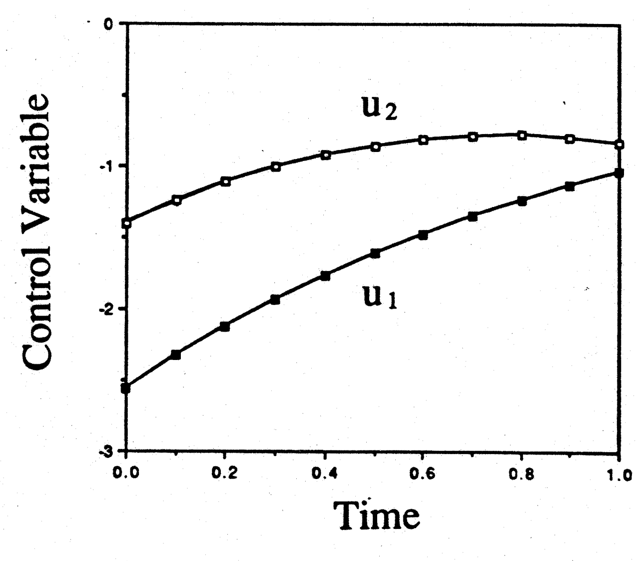

Figure 1b. Control Variable Histories of Example.