Optimal Design of Low Order Controllers Satisfying Sensitivity and Robustness Constraints

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***Abstract*— The set of all stabilizing controllers of a given low order structure that guarantee specifications on the gain margin, phase margin and a bound on the sensitivity corresponds to a region in n-dimensional space defined by the coefficients of the controllers. For several practical criteria defined in the paper it is shown that the optimal controller lies on the surface of that region. Moreover, it is shown how to reduce that region to avoid actuator saturation during operation.**

I. Introduction and Problem Statement

Low order controllers, such as a PID controller with a filtered D term, or cascaded loops consisting of a simple gain outer loop and a filtered PI or PD inner loop, are widely used in industry [1],[2]. Controllers of low-order structure with and without the addition of notch filters represent a large portion of controllers used in mechanical applications.

Consider the open-loop transfer function, $L(s)$, written in general form as

$$L(s) = a\,[P_1(s) + bP_2(s)] \tag{1}$$

for plants $P_1(s)$ and $P_2(s)$ and gains a and b. With $P_1(s) = P(s)$ and $P_2(s) = sP(s)$, $L(s) = (a + bs)P(s)$ represents a PD controller. With $P_1 = (1 + k_i/s)P$ and $P_2(s) = sP(s)H(s)$, $L(s)$ corresponds to a filtered PID controller.

It is shown in [3] that gain and phase margin conditions are guaranteed by the following condition on the closed-loop sensitivity,

$$\left|\frac{1}{1 + kL(s)}\right| \le M \quad \text{for } s = j\omega,\ \forall\omega \ge 0,\ k \in [1, K], \tag{2}$$

where the sensitivity bound $M > 1$ and the gain uncertainty of the plant, k, is in the interval $[1, K]$. The equality of (2) guarantees the following lower bounds on the gain margin, GM, and phase margin, PM,

$$\begin{aligned} GM &= 20\log_{10}(K) + 20\log_{10}\left(\frac{M}{M-1}\right) \\ PM &= 2\arcsin\left(\frac{1}{2M}\right). \end{aligned} \tag{3}$$

It is also shown in [3] how to determine the (a, b) values for which the closed-loop system is stable and (2) is satisfied, starting with $K = 1$, and a single plant pair $(P_1(s), P_2(s))$. Substituting (1) into (2) gives,

$$U + a(U_1 + bU_2) + a^2(V_1 + bV_2 + b^2V_3) \ge 0 \tag{4}$$

$\forall\omega \ge 0$ where

$$\begin{aligned} U &= 1 - 1/M^2 \\ U_1 &= 2\cdot\mathrm{Real}(P_1),\ U_2 = 2\cdot\mathrm{Real}(P_2) \\ V_1 &= |P_1|^2,\ V_2 = 2\cdot\mathrm{Real}(P_1P_2^*),\ V_3 = |P_2|^2. \end{aligned}$$

For an (a, b) pair which is on the boundary of the domain of the allowed (a, b) values, there exists ω such that (4) is an equality. Since at that particular ω (4) is minimum, its derivative (with respect to ω) at the same ω is zero. Thus,

$$\dot{U}_1 + b\dot{U}_2 + a(\dot{V}_1 + b\dot{V}_2 + b^2\dot{V}_3) = 0. \tag{5}$$

Solving (5) for a gives

$$a = -\frac{\dot{U}_1 + b\dot{U}_2}{\dot{V}_1 + b\dot{V}_2 + b^2\dot{V}_3}. \tag{6}$$

Substituting (6) into the equality of (4) gives a fourth-order equation for b whose coefficients are functions of ω,

$$x_4b^4 + x_3b^3 + x_2b^2 + x_1b + x_0 = 0\ , \tag{7}$$

where

$$\begin{aligned} x_4 &= U\dot{V}_3^2 + \dot{U}_2^2V_3 - \dot{U}_2U_2\dot{V}_3 \\ x_3 &= (-U_2\dot{V}_3 + 2\dot{U}_2V_3)\dot{U}_1 - \dot{U}_2U_1\dot{V}_3 \\ &\quad +2U\dot{V}_2\dot{V}_3 + \dot{U}_2^2V_2 - \dot{U}_2U_2\dot{V}_2 \\ x_2 &= \dot{U}_1^2V_3 + (2\dot{U}_2V_2 - U_2\dot{V}_2 - U_1\dot{V}_3)\dot{U}_1 + \dot{U}_2^2V_1 \\ &\quad -\dot{U}_2U_1\dot{V}_2 + U\dot{V}_2^2 + 2U\dot{V}_1\dot{V}_3 - \dot{U}_2U_2\dot{V}_1 \\ x_1 &= \dot{U}_1^2V_2 + (-U_1\dot{V}_2 - U_2\dot{V}_1 + 2\dot{U}_2V_1)\dot{U}_1 \\ &\quad +2U\dot{V}_1\dot{V}_2 - \dot{U}_2U_1\dot{V}_1 \\ x_0 &= -\dot{U}_1U_1\dot{V}_1 + U\dot{V}_1^2 + \dot{U}_1^2V_1\ . \end{aligned}$$

The boundary of allowed (a, b) values for a given M can be calculated as follows: For a given ω solve (7) for b. Noting that b has four solutions (for a given ω), pick the positive real solution(s) and use (6) to find their corresponding a. Select the (a, b) pairs for which the resulting closed-loop system is stable and (2) is satisfied. Searching over a range of frequencies, ω, gives two vectors of ω, $(a(\omega), b(\omega))$ which lie on the boundary of the allowed (a, b) values. This algorithm can be extended to include plant gain uncertainty in an interval [3].

The approach can be extended to design PID controllers with a filter $H(s)$ on the D-term, i.e.,

$$\begin{aligned} L(s) &= a\left[\left(1+\frac{k_i}{s}\right)P_1(s)+bP_2(s)H(s)\right] \\ &= a\left[\tilde{P}_1(s)+b\tilde{P}_2(s)\right] . \end{aligned} \tag{8}$$

It involves searching on k_i and $H(s)$ as follows: Choose the structure of the filter $H(s)$, for example, $H(s) = \frac{p}{s+p}$ or $H(s) = \frac{p^2}{s^2+ps+p^2}$, fix k_i and p and calculate its boundary (a, b). It is shown in [3] how to pick the intervals of p and k_i for calculating (a, b).

The extension to include notch filters operating on the signal at the plant input, that is,

$$L(s) = a\left[\left(1+\frac{k_i}{s}\right)P_1(s)+bP_2(s)H(s)\right]N(s),$$

where $N(s)$ are notch filters, is straightforward.

The region of all controllers whose structure is given by (8) is defined as all (a, b) pairs for all possible (k_i, p) pairs. The problem considered here is under what conditions the optimal controller lies on the hyper-plane of that region and not on an internal point. This is a significant question because it shrinks the space in the search for the optimal controller.

II. Optimization

For a controller of the structure of (8), the answer to the question of which (a, b) pair is optimal clearly depends on the optimization criterion. Seron and Goodwin [4] note that "In general, the process noise spectrum is typically concentrated at low frequencies, while the measurement noise spectrum is typically more significant at high frequencies." The conclusion is that an optimal controller is a result of weighting the performance at low frequencies and of noise at high frequencies. Since the high frequency noise is proportional to ab and low frequency performance to $1/a$, a practical optimal criterion can be

$$J = \alpha(1/a) + \beta(ab) .$$

Lemma 2.1: Let $J = f(a) + g(ab)$ where f, g are strongly monotonic be a functional operating on the (a, b) domain defined by (2). Then the minimum of J on the domain (a, b) will be achieved on a point for which (2) is an equality (that is, on the boundary of all (a, b) satisfying (2)).

From lemma 2.1 the optimal solution must lie on the boundary of the (a, b) domain. When β is small enough or zero (meaning the sensor noise is neglected), the optimal solution is the maximum possible a (qualitatively best sensitivity solution).

Another practical optimal criterion is based on the observation that the noise should be below a certain level. The optimal criterion will then be the (a, b) pair for which a is maximum and ab is below a given level. Another required controller can be the one whose high frequency noise, ab, is the lowest while the low frequency sensitivity, a, is better than a given figure. Again from Lemma 2.1 for this optimal criterion the optimal solution lies on the boundary.

Lemma 2.1 suits a more general case. It shows that for the criterion $J = f(a) + g(ab)$ where $f(a)$ and $g(ab)$ are monotonic the optimal solution lies on the boundary of the (a, b) domain.

Proof (by contradiction): Let a_0, b_0 be a point for which $J = J_0$ is minimum and (2) is an inequality. Therefore, there is an open ball with radius ϵ around a_0, b_0 for which (2) is still an inequality. However at one of the following points $a_0 \pm \epsilon/2, b_0 \pm \epsilon/2$, $J > J_0$ because f, g are strongly monotonic. Thus a_0, b_0 is not a minimum point.

In many applications the criterion is to find a controller whose crossover frequency is given, where the optimality criterion is a controller whose low frequency sensitivity, $1/a$, is minimum and/or high frequency noise is minimum. Under reasonable conditions, specified in the following lemma, this optimal solution also lies on the boundary of the (a, b) domain.

Lemma 2.2: Let (a, b) denote the set of all a, b pairs satisfying (2) and (a_ω, b_ω) denote the subset of (a, b) whose crossover frequency is ω, that is all (a_ω, b_ω) pairs in (a, b) satisfying

$$|a_\omega(P_1(j\omega) + b_\omega P_2(j\omega))| = 1. \tag{9}$$

Assuming a solution with crossover frequency ω exists, then

1) if $|P_1(j\omega) + b_\omega P_2(j\omega)|$ is an increasing function of b_ω, then the maximum value of a_ω satisfies equality of (2);
2) the minimum value of $a_\omega b_\omega$ satisfies the equality of (2).

Proof of 1 (by contradiction): Let a_ω, b_ω satisfy (9), with a_ω being maximum and for which a_ω, b_ω (2) is an inequality. Then because of the monotonic property there exist $a > a_\omega$ and $b < b_\omega$ in (a, b) satisfying (9), which is a contradiction.
Proof of 2 (by contradiction): (9) can be written in the form

$$\begin{aligned} a_\omega &= -a_\omega b_\omega(\mathrm{real}(P_2/P_1)) \pm \\ &\quad \sqrt{|1/P_1|^2 - (a_\omega b_\omega)^2(\mathrm{imag}(P_2/P_1))^2} . \end{aligned}$$

If (a_ω, b_ω) is a solution of (9) and is not an equality of (2), there exists another solution, (a, b), in any small enough neighborhood of (a_ω, b_ω) for which $ab < a_\omega b_\omega$. This is a contradiction.

Example 1: $C(s) = a(1 + bs)$. Condition 1 of lemma 2.2 is satisfied because

$$|P_1(j\omega) + b_\omega P_2(j\omega)| = \sqrt{(1 + b^2\omega^2)}|P(j\omega)|$$

is an increasing function of b.

Example 2: $C(s) = a(1 + bH(s))$. Condition 1 of lemma 2.2 is satisfied if $|1 + bH(j\omega)||P(j\omega)|$ is an increasing function of b. This will be true if the phase of $H(j\omega)$ is in the interval $[-\pi/2, \pi/2]$. This condition can be relaxed to $b|H(j\omega)| > -\mathrm{real}(H(j\omega))$. Important $H(s)$ filters with phase in the interval $[-\pi/2, \pi/2]$ are: (i) $H = s/(1+s/p)$, that is, a low pass filter of order 1 on the derivative term, and (ii) $H = s/(1 + 2\xi s/\omega + s^2/\omega^2)$, that is, a low pass filter of order 2 on the derivative.

A. Constraint Optimization

Other practical optimal criterion would include any optimal criterion subject to hard limitation constraints on the control efforts to avoid actuator saturation. Such conditions will shrink the (a, b) domain. We show next how to extract these sub-domains. Two possibilities are addressed, one in which there exist two independent actuators, which are the inputs to P_1 and P_2, and the second with a single actuator. The two theorems given above are replaced by the following while the proof remains the same.

Lemma 2.3: Let $J = f(a) + g(ab)$ where f, g are strongly monotonic be a functional operating on the (a, b) domain defined by (2) intersected with another domain D. Then the minimum of J on that domain will be achieved on a point for which (2) is an equality or a point on the boundary of D.

Lemma 2.4: Let (a, b) denote the set of all (a, b) pairs satisfying (2) intersected with another domain D and (a_ω, b_ω) denote a subset whose crossover frequency is ω, that is all (a_ω, b_ω) pairs in (a, b) satisfying

$$|a_\omega(P_1(j\omega) + b_\omega P_2(j\omega))| = 1. \tag{10}$$

Assuming a solution with crossover frequency ω exists, then

1) if $|P_1(j\omega) + b_\omega P_2(j\omega)|$ is an increasing function of b_ω, then the maximum value of a_ω satisfies equality of (2) or is on the boundary of D;
2) the minimum value of $a_\omega b_\omega$ satisfies equality of (2) or is on the boundary of D.

Consider the discrete version of the controller, that is, $C(z) = a + ab(1 - z^{-1})$. It is required to find a controller whose output is $ae(k)+ab(e(k)-e(k-1))$ and is bounded where $e(k)$ is the input to the controller at time k. Given the constants u_{min} and u_{max}, the controller should satisfy

$$u_{min} \le ae(k) + ab(e(k) - e(k-1)) \le u_{max}, \tag{11}$$

which, geometrically, is all (a, ab) pairs between and on two lines in the $a - ab$ plane. Now assume that the (a, ab) pair is updated at each sampling time. (A question not addressed here is how fast it can be updated without sacrificing performance or even stability.) Then the boundary of the allowed (a, ab) pairs will be the intersection of the original (a, ab) domain and the domain dictated by (11).

III. Conclusions

This paper examines an analytically-based algorithm for finding low order controllers that satisfy closed loop gain and phase margin constraints and a bound on the sensitivity. The application of the algorithm gives a dense set of controllers that lie on the hyper-plane of all possible controllers. The paper gives practical criteria for which the optimal controller is a member of this hyper-plane. In addition, it shows how to extend the hyper-plane to include actuator saturation constraint.

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