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A FOURIER-BASED METHOD FOR THE SUBOPTIMAL CONTROL OF NONLINEAR DYNAMICAL SYSTEMS

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Abstract

Trajectories (of generalized coordinates and control variables) of lumped-parameter, dynamical systems are simulated using a suboptimal control approach. The approach is a Fourier-based approximation technique that converts an optimal control problem into a nonlinear programming problem that can be solved using well-developed optimization algorithms. Results of simulation studies show that the approach can be used to predict suboptimal trajectories of high dimension, nonlinear systems with constraints.

1 Introduction

The application of optimal control theories to many physical systems, including robotic manipulators and structural systems, has been driven by the demand for systems of high performance, coupled with the availability of ever more powerful digital computers. However, the implementation of optimal control theories, especially for high dimension, nonlinear systems, has been hindered by numerical difficulties (as discussed below.)

In the classical development, the optimal control of unconstrained, linear systems with quadratic performance indices can be determined via the solution of the Riccati equation. However, for general dynamical systems (*i.e.*, systems with nonlinearities and/or constraints) analytically tractable approaches, such as offered by the Riccati equation, are usually not available. For the general case, the variational method of optimal control theory must be applied to determine a set of necessary conditions for optimality [1-3]. These necessary conditions lead to a (generally nonlinear) two-point boundary-value problem (2PBVP) of state-costate equations that must be solved to find an explicit expression for the optimal control.

There are two principal difficulties in solving the 2PBVP. First, the boundary conditions for the state-costate equations are given at separate ends (initial conditions are known for state equations, terminal condition are known for costate equations). As a result, the state-costate equations often must be solved by iterative integration algorithms. Second, different types of constraints or penalty functions on the terminal states and/or terminal time lead to different types of 2PBVP which, in general, require different numerical solution methods. As a result, there is usually a significant demand on programming effort.

Various numerical techniques, represented by gradient-based methods and dynamic programming methods, have been proposed to help overcome the above problems. Gradient methods include algorithms such as steepest-descent and variation of extremal (Newton type) techniques. A drawback of these methods is their sensitivity to computational errors, which often leads to their failure in solving high order optimal control problems. Dynamic programming represents another class of methods. However, its application is often limited due to dimensionality problems [4].

The approach employed in this paper draws upon the power of nonlinear programming to determine suboptimal (*i.e.*, near optimal) trajectories of general dynamical systems.

2 Methodology

Consider a dynamical system represented by a coupled set of nonlinear, second-order differential equations:

$$\underline{F}[\underline{X}(t),\ \dot{\underline{X}}(t),\ \ddot{\underline{X}}(t),\ t] \ = \ \underline{U}(t) \tag{1}$$

where $\underline{X}(t)$ is an N dimensional vector of generalized coordinates, $\underline{U}(t)$ is an N dimensional vector of control variables, and superscript dot represents differentiation with respect to time, t. Many dynamical systems have the same number of generalized coordinates and control variables, and can be represented by equation (1). Typically, initial conditions on $\underline{X}(0)$ and $\dot{\underline{X}}(0)$ are specified.

The optimal trajectory, $\underline{X}^*(t)$, $\dot{\underline{X}}^*(t)$, and $\ddot{\underline{X}}^*(t)$, is defined as the admissible trajectory that minimizes the performance index, J,

$$J \ = \ E[\underline{X}(t_f),\ \dot{\underline{X}}(t_f),\ t_f] \ + \int_0^{t_f} G[\underline{X}(t),\ \dot{\underline{X}}(t),\ \underline{U}(t),\ t]\,dt \tag{2}$$

subject to constraints, where $[\,0,\ t_f]$ is the time interval of the trajectory, t_f being the terminal time. Two different types of constraints can be identified, *i.e.*, inequality constraints:

$$\underline{C}[\underline{X}(t),\ \dot{\underline{X}}(t),\ \underline{U}(t)] \geq \underline{0} \tag{3}$$

and terminal constraints:

$$\underline{D}[\underline{X}(t_f),\ \dot{\underline{X}}(t_f),\ \ddot{\underline{X}}(t_f),\ t_f] = \underline{0} \tag{4}$$

It is assumed that (i) an optimal trajectory exists and is unique, and (ii) $\underline{X}^*(t)$ and $\dot{\underline{X}}^*(t)$ of the optimal trajectory are continuous and $\ddot{\underline{X}}^*(t)$ is piece-wise differentiable.

2.1 Fourier-Based Approach

Without resorting to variational methods, Yen and Nagurka [5] have proposed a Fourier-based approach to generate the suboptimal trajectories of dynamical systems. The basic idea is to approximate each generalized coordinate by the sum an auxiliary polynomial and finite terms of a Fourier-type series. For example, the i-th generalized coordinate is represented by

$$X_i(t) = P_i(t) + \sum_{m=1}^{M} a_{im} \cos\frac{2m\pi t}{t_f} + \sum_{m=1}^{M} b_{im} \sin\frac{2m\pi t}{t_f} \tag{5}$$

where M is the number of terms included in the Fourier-type series, and $P_i(t)$ is a fifth-order auxiliary polynomial

$$\begin{aligned} P_i(t) = {} & [-6(P_{i0}-P_{if}) - 3(\dot{P}_{i0}+\dot{P}_{if})t_f - \tfrac{1}{2}(\ddot{P}_{i0}-\ddot{P}_{if})t_f^2]\tau^5 \\ & + [15(P_{i0}-P_{if}) + (8\dot{P}_{i0}+7\dot{P}_{if})t_f + (\tfrac{3}{2}\ddot{P}_{i0}-\ddot{P}_{if})t_f^2]\tau^4 \\ & + [-10(P_{i0}-P_{if}) - (6\dot{P}_{i0}+4\dot{P}_{if})t_f - \tfrac{1}{2}(3\ddot{P}_{i0}-\ddot{P}_{if})t_f^2]\tau^3 \\ & + [\tfrac{1}{2}\ddot{P}_{i0}t_f^2]\tau^2 + [\dot{P}_{i0}t_f]\tau + P_{i0} \end{aligned} \tag{6}$$

where $\tau = t/t_f$ and $P_{i0} = P_i(0)$, $P_{if} = P_i(t_f)$, and similarly for their time derivatives. The boundary values for the auxiliary polynomial are determined from the boundary conditions of the generalized coordinates and their rates applied to equation (5) as follows:

$$P_{i0} = X_{i0} - \sum_{m=1}^{M} a_{im}\ , \quad P_{if} = X_{if} - \sum_{m=1}^{M} a_{im} \tag{7a,b}$$

$$\dot{P}_{i0} = \dot{X}_{i0} + 2\sum_{m=1}^{M} \frac{m\pi}{t_f} b_{im} \quad , \quad \dot{P}_{if} = \dot{X}_{if} + 2\sum_{m=1}^{M} \frac{m\pi}{t_f} b_{im} \tag{8a,b}$$

$$\ddot{P}_{i0} = \ddot{X}_{i0} + 4\sum_{m=1}^{M} \frac{m^2\pi^2}{t_f^2} a_{im} \quad , \quad \ddot{P}_{if} = \ddot{X}_{if} + 4\sum_{m=1}^{M} \frac{m^2\pi^2}{t_f^2} a_{im} \tag{9a,b}$$

where $X_{i0} = X_i(0)$, $X_{if} = X_i(t_f)$, and similarly for the corresponding time derivatives. In general, $\ddot{X}_{i0}$ is free. Depending upon equation (4), X_{if}, $\dot{X}_{if}$, and $\ddot{X}_{if}$ may be free or constrained.

The free boundary conditions and the coefficients of Fourier-type series of all generalized coordinates, as well as the terminal time (if it is not fixed), are free variables. These free variables are adjusted simultaneously by a nonlinear programming method such that the performance index is minimized without violating the constraints (3). In the simulation studies of this paper, a nonlinear Simplex algorithm implemented on a PC was used to solve the nonlinear programming problem.

This method is a suboptimal (*i.e.*, near optimal) approach that becomes truly optimal only as the number of terms in the Fourier-type series used to represent the generalized coordinate trajectories approaches infinity. As more terms are included, the suboptimal performance index converges to its minimum value, *i.e.*, the optimal performance index.

It is possible to relax the assumption above that the number of control variables is equal to the number of degrees of freedom of the system. There are two possible exceptions to this assumption, as follows:

1. The number of control variables is greater than the number of generalized coordinates. A system with this characteristic is called "redundant." To apply the Fourier-based method to a redundant system, the time history of each redundant control variable is represented by a finite term Fourier-type series. The coefficients are then determined together with the other free variables to minimize the performance index.

2. The number of control variables is less than the number of generalized coordinates. A system of this type usually can be partitioned into a subsystem:

$$\underline{F}_\alpha[\underline{X}(t),\ \dot{\underline{X}}(t),\ \ddot{\underline{X}}(t),\ t] = \underline{U}(t) \tag{10}$$

involving a vector of control variables $\underline{U}(t)$ of dimension L, and a complementary subsystem

$$\underline{F}_\beta[\underline{X}(t),\ \dot{\underline{X}}(t),\ \ddot{\underline{X}}(t),\ t] = \underline{0} \tag{11}$$

that is free of control variables. (This is the case of Example 2 below.) From equations

(10) and (11) a set of L generalized coordinates is chosen such that the remaining generalized coordinates (*i.e.*, N - L of these variables) can be calculated from equation (11) (*e.g.*, by integration). Each of the L generalized coordinates is approximated by a Fourier-type function (5), and the control variables are computed from equation (10). Note that the terminal conditions still must be satisfied simultaneously by equations (10) and (11).

3 Simulation Studies

3.1 Example 1

This example is adapted from [2, Case A, pp. 198-202] . The dynamical system is a linear, second order system described by:

$$\ddot{X}(t) + X(t) = U(t) \tag{12}$$

with initial conditions

$$X(0) = 0 \quad , \quad \dot{X}(0) = 0 \tag{13}$$

In [2] equation (12) is written as a system of two first order state equations in terms of $X_1(t) = X(t)$ and $X_2(t) = \dot{X}(t)$.

The system is to be controlled such that its control effort, defined by the following performance index, is minimized.

$$J = \frac{1}{2}\int_0^2 U^2(t)\,dt \tag{14}$$

The terminal constraints are

$$X_1(2) = 5 \quad , \quad X_2(2) = 2 \tag{15}$$

The admissible states and control variables are not bounded. The closed-form optimal solution (derived in [2]) is

$$X_1^*(t) = 7.289t - 6.103 + 6.696e^{-t} - 0.593e^{t} \tag{16}$$

$$X_2^*(t) = 7.289 - 6.696e^{-t} - 0.593e^{t} \tag{17}$$

This problem was solved using the Fourier-based approach, *i.e.*, $X_1(t)$ was represented by equation (5) with a one-term ($M = 1$) Fourier-type series. The performance

Table 1: Suboptimal Values for Example 1 and Comparison to Optimal Performance Index.

Suboptimal Values	a	-1.9564395E-03
	b	1.4421728E-03
	$\dot{X}_2(0)$	6.1025137E+00
	$X_1(2)$	5.0000000E+00
	$X_2(2)$	2.0000000E+00
	$\dot{X}_2(2)$	-3.4798053E+00
	J	1.675072E+01
	Optimal J	1.674543E+01
	% Error	< 0.04%

index was evaluated by means of Simpson's composite integral formula with a step size of 1/30 (consistent unit of time).

The optimal values of the free variables and the performance index are listed in Table 1. From the table it can be seen that the suboptimal solution achieves high accuracy, although only a one-term series was used.

The time histories of $U(t)$ and $X_1(t)$ of the optimal solution (from the closed-form expressions) and the suboptimal solution are plotted in Figures 1a and 1b. The time history of the auxiliary polynomial is also plotted in Figure 1b. These figures show that the optimal and suboptimal solutions are essentially coincident. Furthermore, Figure 1b shows that the auxiliary polynomial approximates the optimal (and suboptimal) solutions. This phenomenon is typically exhibited for smooth optimal trajectories, which occur for most optimal control problems.

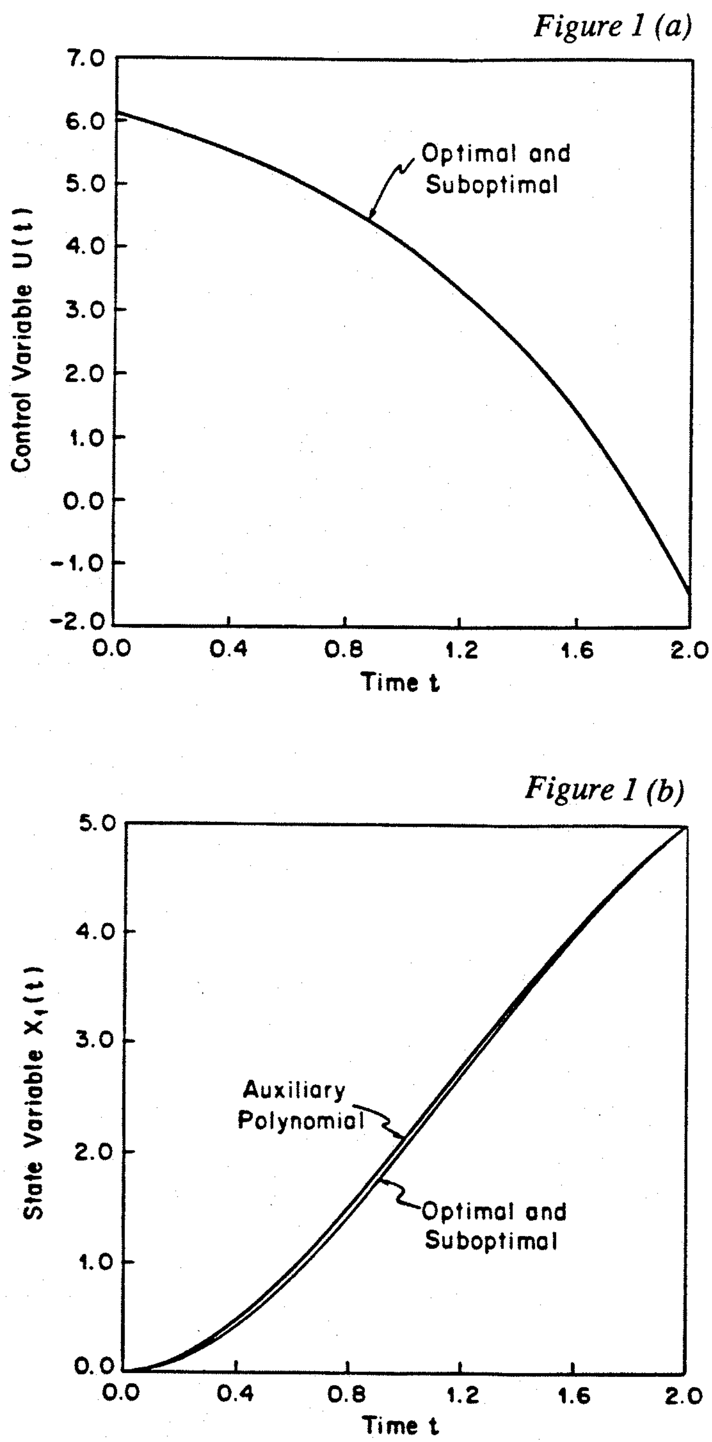

Figure 1: History of (a) Control Variable U, and (b) State Variable X_1 and Auxiliary Polynomial P for **Example 1.**

3.2 Example 2

This example is adapted from [2, pp. 405-407]. The state equations of the system are highly nonlinear and coupled:

$$\dot{X}_1(t)=-2[X_1(t)+0.25]+[X_2(t)+0.5]\,exp[\frac{25X_1(t)}{X_1(t)+2}]-[X_1(t)+0.25]\,U(t) \quad (18)$$

$$\dot{X}_2(t)=0.5-X_2(t)-[X_2(t)+0.5]\,exp[\frac{25X_1(t)}{X_1(t)+2}] \quad (19)$$

and are derived from a model of a continuous stirred-tank chemical reactor described in [7]. (Here, $exp\ [\gamma]\ = e^{\gamma}$.) The initial conditions are:

$$X_1(0)=0.05\ ,\qquad X_2(0)=0.00 \quad (20a,b)$$

The performance index is

$$J=\int_0^{0.78}[X_1^2(t)+X_2^2(t)]\,dt \quad (21)$$

Two sets of constraints are imposed. The first set bounds the control as follows:

$$-1.0 \le U(t) \le 1.0 \qquad for \quad t \in \ [0\ \ 0.78] \quad (22)$$

The second set of constraints fixes the terminal state as follows:

$$X_1(0.78)=0.0\ ,\qquad X_2(0.78)=0.0 \quad (23)$$

This example addresses a system with more generalized coordinates (X_1 and X_2) than control variables (U). To solve this problem, X_2 is approximated by the sum of a fifth order polynomial and a two term Fourier-type series. $\dot{X}_2\ (0.78)$ is known from the second state equation with $X_1\ (0.78)\ =\ X_2\ (0.78)\ =\ 0.0$, whereas $\dot{X}_1\ (0.78)$ is free.

The control variable is a function of $X_1(t)$, $\dot{X}_1(t)$, and $X_2(t)$. $X_1(t)$ can be obtained explicitly in terms of the approximation for $X_2(t)$ and $\dot{X}_2(t)$ directly from the second state equation. $\dot{X}_1(t)$ can then be found by direct differentiation. Thus, the control variable can be determined, its constraints can be checked, and the performance index can be evaluated.

In [2] this problem was solved by the gradient projection method. The values of the performance index as well as the terminal values of the generalized coordinates are given in Table 2. The suboptimal values agree closely to the optimal results.

Table 2: Comparison of Results for Example 2.

METHOD	J	$X_1(0.78)$	$X_2(0.78)$
Gradient Projection	0.0022	-6.167E-06	-0.631E-06
Suboptimal Method	0.0021	< 1.0E-09	< 1.0E-09

The time history of the control variable and the history of the generalized coordinates are plotted in Figures 2a and 2b, respectively. The control variable determined from the gradient projection method shows two nondifferentiable points at the boundaries of the saturation regions of the control variable. In contrast, the suboptimal control is differentiable for the entire time, although it approaches the regions of saturation associated with the optimal solution. Despite the differences in the control histories, the generalized coordinate histories for the two methods are close.

This example demonstrates the application of the Fourier-based method for the solution of a constrained, nonlinear, optimal control problem. The complete satisfaction of the final boundary conditions as well as the close match of the performance index to that of an independent method (the gradient projection method) verify the method's effectiveness.

4 Discussion

In the Fourier-based method, the derivatives of the generalized coordinates are obtained by analytical differentiation of equation (5), and thus the system equations (1) are treated as algebraic equations in evaluating the control variables. As such, the Fourier-based method is an *inverse dynamic* approach. It finds the optimal solution by adjusting the trajectory itself and the control variables become the "outputs."

In contrast, gradient-based methods are *direct dynamic* approaches. An important consequence of this distinction relates to numerical error. The direct dynamic approach involves integration of differential equations in which computational errors often tend to propagate. In fact, it is the accumulation of errors that frequently leads to failure in solving optimal control problems for high dimension systems. In an inverse dynamic approach, the control variables are obtained by straightforward algebra. The only significant computational error is due to the numerical integration of the performance index. Usually, this error can be controlled and estimated easily.

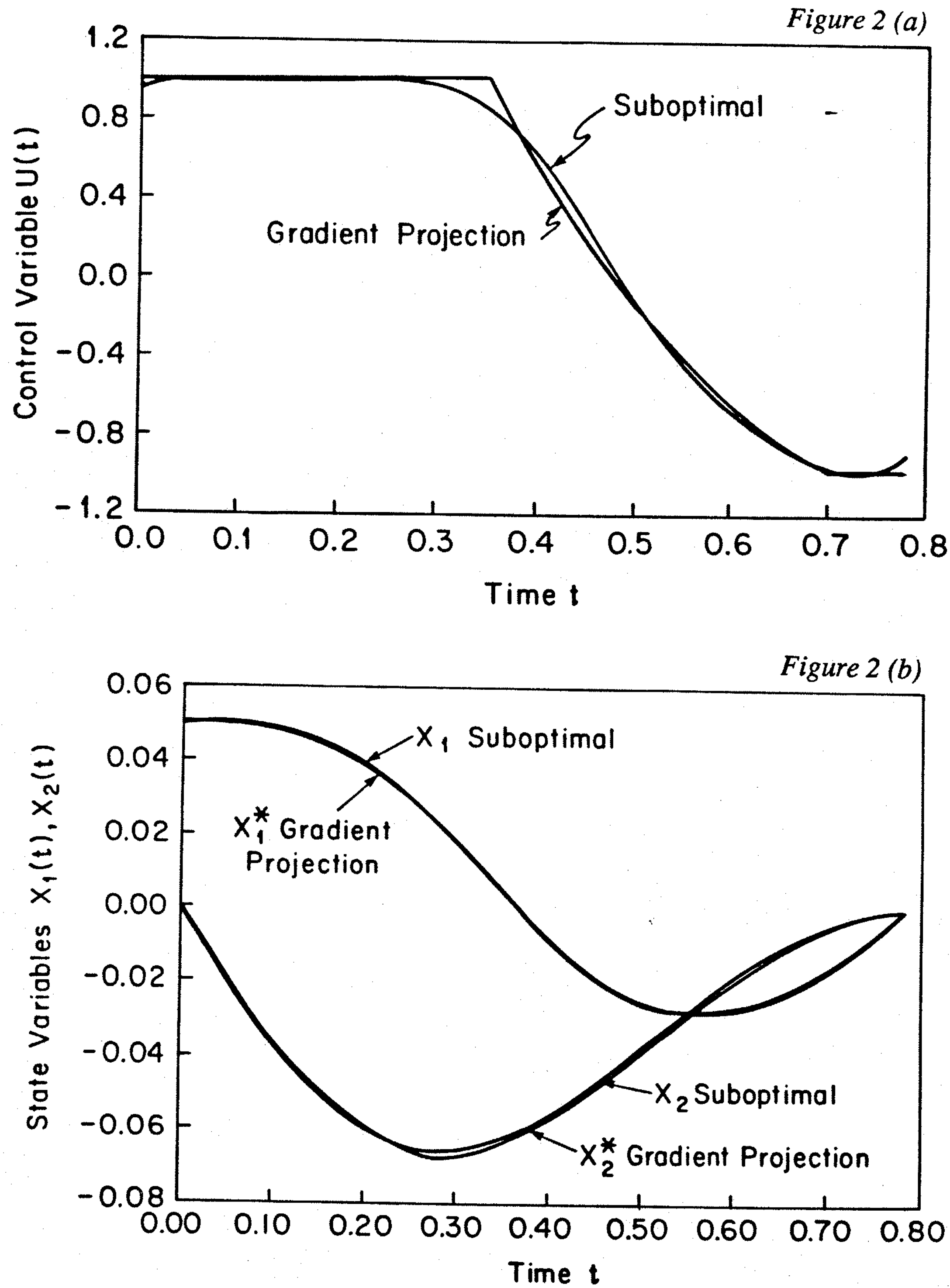

Figure 2: History of (a) Control Variable U, and (b) State Variables X_1 and X_2 for **Example 2.**

The Fourier-based method offers efficient utilization of computer memory. It is distinct from dynamic programming methods in the way that it approximates the time history of the trajectory. Dynamic programming methods typically divide the time history of the generalized coordinates into a finite number of intervals. Optimality is achieved by finding the optimal values of the generalized coordinates at each time interval. In contrast, the Fourier-based method approximates the time history of each generalized coordinate by a single function. Optimality is found by adjusting a (typically) much smaller number of parameters for the function(s). The computer memory requirement is therefore greatly reduced.

The quality of the Fourier-based suboptimal solution can be assessed by checking if it satisfies the necessary conditions for optimality which are derived by variational techniques. In practice, this verification can be done by substituting the suboptimal solution into an appropriate, standard, optimal control algorithm and determining if the termination criterion of the selected algorithm can be satisfied.

An alternative empirical approach to verify the optimality is to append another term of the Fourier-type series to the previous solution and re-execute the algorithm. Additional terms can be added, on a term by term basis, until the value of the performance index converges, indicating that the optimal solution has been reached.

Finally, it is worth noting that simulation results [5] suggest that the Fourier-based method is not suitable for exact solution of bang-bang control problems. However, the suboptimal solution provides a continuous and therefore physically implementable control law, whereas bang-bang control is a mathematical idealization that can only be approached in practice. The accuracy of the method can be improved by increasing the number of terms of the series, although this increases the computational cost since the speed of convergence for bang-bang control problems is slow.

5 Summary

This paper describes a conceptually simple method for producing suboptimal trajectories of dynamical systems represented by deterministic, lumped-parameter models. The method is based on a Fourier-type series approximation of each generalized coordinate that converts the optimal control problem into an algebraic nonlinear programming problem. Due to its inverse dynamic nature, the method avoids many of the numerical difficulties typically encountered in solving standard optimal control problems.

The results of computer simulation studies demonstrate that the Fourier-based approach (i) is easy to implement, requiring minimum analytical and programming effort, (ii) is capable of handling various types of constraints, and (iii) is quite effective for solving non-bang-bang type control problems. Perhaps the most significant advantage of the approach is its ability to tackle linear as well as nonlinear high dimension systems.

6 References

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