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Fourier-Based Optimal Control of Nonlinear Dynamic Systems

A method for generating near optimal trajectories of linear and nonlinear dynamic systems, represented by deterministic, lumped-parameter models, is proposed. The method is based on a Fourier series approximation of each generalized coordinate that converts the optimal control problem into an algebraic nonlinear programming problem. Due to its inverse dynamic nature, the method avoids many of the numerical difficulties typically encountered in solving standard optimal control problems. Furthermore, the method is easy to implement, capable of handling various types of constraints, and quite effective for solving non-bang-bang type control problems. The results of computer simulation studies compare favorably to optimal solutions obtained by closed-form analyses and/or by other numerical schemes.

Introduction

Optimal control theories are playing an increasingly important role in the design of modern systems. This role has been spurred by the demand for systems of high performance, especially as more powerful computers become readily available. However, serious analytical and numerical difficulties, such as accumulation of roundoff and truncation errors, need to be overcome before optimal control approaches will find widespread practical implementation, especially for high order, nonlinear systems.

In the classical development, the variational method of optimal control theory, which typically consists of the calculus of variations and Pontryagin's methods, can be used to derive a set of necessary conditions that must be satisfied by an optimal control law and its associated state-costate equations (sometimes called state-adjoint equations) (Athans and Falb, 1966; Kirk, 1970; Sage and White, 1977). These necessary conditions of optimality lead to a (generally nonlinear) two-point boundary-value problem (2PBVP) that must be solved to determine an explicit expression for the optimal control.

There are two principal difficulties in solving the 2PBVP. First, the boundary conditions for the state-costate equations are known at separate ends (initial conditions are given for state equations, terminal conditions are given for costate equations). As a result, the state-costate equations often must be solved by iterative integration algorithms. Second, different types of constraints or penalty functions on the terminal states and/or terminal time lead to different types of 2PBVP. In general, the solution of these distinct problems requires different numerical methods, with the concomitant drawback of increased programming effort.

Various numerical techniques, such as gradient-based methods, have been proposed to overcome the above problems. Gradient methods include algorithms such as steepest-descent and variation of extremal (Newton-type) techniques. For these methods, a termination criterion is usually found by trial-and-error and convergence very often depends on the initial guess. A more serious drawback of gradient methods is their sensitivity to computational errors, which often leads to their failure in solving high order optimal control problems.

Dynamic programming represents another class of methods. From a computational perspective, dynamic programming, which is based on the principle of optimality, is particularly well suited for solving allocation problems and sequential decision problems. In contrast, it is not well suited for solving optimal control problems, where it is necessary to quantize the state and control variables into a finite number of admissible values. The problem is that the computer memory requirement becomes prohibitively large for high order systems. Bellman (1957), the father of dynamic programming, called this problem the "curse of dimensionality." In addition, dynamic programming is not effective in handling problems with free terminal time. Many strategies, such as quasi-linearization, grid-manipulation, etc., have been proposed to solve these difficulties, but in general they are successful only in limited cases. In summary, the solution of optimal control problems involving high order, nonlinear systems remains a challenge.

In the area of optimization design, on the other hand, well-developed nonlinear programming techniques such as Powell's method and the variable metric method have been applied successfully in many engineering applications. There exists a number of general purpose nonlinear programming computer codes (e.g., Belegundu and Arora, 1985) which design engineers have used to determine optimal equipment specifications, optimal operating conditions, optimal features of plant expansion, etc.

The approach proposed in this paper draws upon the power of nonlinear programming to determine optimal trajectories of high order, nonlinear systems. Central to the idea is the approximation of the time response of each generalized coor-

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dinate by the sum of a polynomial in time and finite terms of a Fourier-type series. In this way, it is possible to formulate the problem as an algebraic nonlinear programming problem with the coefficients of the Fourier-type functions, the free boundary conditions, and the terminal time, if it is free, as the variables adjusted in minimizing the performance index, i.e., the objective function of the optimization. An important benefit of recasting the problem as a nonlinear programming problem is that it eliminates the requirement of solving a 2PBVP. In contrast to dynamic programming, the proposed method does not require massive computer storage. It thereby offers a streamlined approach for solving general optimal control problems.

Background

Fourier and polynomial approximation techniques are used commonly in engineering. For instance, in the analysis of structures, periodic forcing functions are often expanded in Fourier series enabling the steady-state response of a structural model to be determined. Although theoretically a Fourier series representation of a general periodic forcing function requires an infinite number of terms, in practice the forcing function may often be approximated with sufficient accuracy by a relatively small number of terms as long as the terms close to the resonant frequencies are not neglected.

In planning optimal motions of robotic manipulators, Schmitt et al., (1985) have proposed an approach based on the Raleigh-Ritz scheme. In this approach, the trajectory of each degree of freedom of the robotic manipulator is approximated by the sum of a sequence of known functions and a third order polynomial whose coefficients are functions of the initial and terminal conditions. The necessary conditions of optimality are obtained by setting the first partial derivatives of the performance index with respect to the coefficients of the known functions equal to zero. This leads to a system of nonlinear algebraic equations which can be solved by any of several numerical techniques.

In (Schmitt et al., 1985) the Raleigh-Ritz solution does not necessarily converge to the optimal solution (since the approximation functions are not all functionally complete) and the approach does not handle non-fixed terminal state problems effectively. Furthermore, the inclusion of system constraints is not discussed and the approach is not generalized for application to dynamic systems other than robotic manipulators.

This paper develops a Fourier-based method which generalizes the approximation technique used in (Schmitt et al., 1985) to solve for the optimal trajectories of nonlinear dynamic systems. By implementing a nonlinear programming algorithm, the requirement of finding a closed-form expression for the necessary and sufficient conditions of optimality is eliminated. This minimizes the requisite analytical work. As demonstrated by examples, the method can be used to generate optimal trajectories of free, fixed, and constrained optimal control problems.

A specialization of this approach to linear structural systems with quadratic performance indices has been developed (Yen and Nagurka, 1988). For such systems, the necessary and sufficient conditions of optimality can be derived as a system of linear algebraic equations, which can readily be solved. The results of simulation studies suggest that, in comparison to a standard Riccati equation solver, this Fourier-based method is more efficient in computation and more robust in handling heavily penalized terminal states while maintaining satisfactory accuracy.

Methodology

Consider an N degree-of-freedom dynamic system described by the following system of differential equations:

$$\mathbf{F}[\mathbf{X}(t),\ \dot{\mathbf{X}}(t),\ \ddot{\mathbf{X}}(t),\ t]=\mathbf{U}(t),$$
$$\mathbf{X}(0)=\mathbf{X}_0,\ \dot{\mathbf{X}}(0)=\dot{\mathbf{X}}_0 \tag{1}$$

where $\mathbf{X}(t)$ is an N dimensional vector of generalized coordinates, $\mathbf{U}(t)$ is an N dimensional vector of control variables, $\mathbf{F}$ is the set of ordinary differential equations used to describe the system behavior, and superscript dot represents differentiation with respect to time, t. Many dynamic systems, including structural systems and robotic manipulators, have the same number of generalized coordinates and control variables, and can be represented by equation (1). Physically, equation (1) represents a dynamic system which is actively controlled, i.e., each degree of freedom is completely controllable regardless of the coupling between generalized coordinates. The control vector can be uniquely determined once $\mathbf{X}(t)$, $\dot{\mathbf{X}}(t)$, and $\ddot{\mathbf{X}}(t)$ are known. (The application of the proposed approach to dynamic systems which cannot be represented by equation (1) is discussed later.)

The optimal trajectory, $\mathbf{X}^*(t)$, $\dot{\mathbf{X}}^*(t)$, and $\ddot{\mathbf{X}}^*(t)$, is defined as the admissible trajectory that minimizes the performance index, J,

$$J=E[\mathbf{X}(t_f),\ \dot{\mathbf{X}}(t_f),\ t_f]+\int_0^{t_f} G[\mathbf{X}(t),\ \dot{\mathbf{X}}(t),\ \mathbf{U}(t),\ t]dt \tag{2}$$

subject to constraints, where $[0,\ t_f]$ is the time interval of the trajectory. Here, E represents the cost associated with the terminal states and G represents the cost associated with the trajectory. Two different types of constraints can be identified, i.e., inequality constraints, $\mathbf{C}$:

$$\mathbf{C}[\mathbf{X}(t),\ \dot{\mathbf{X}}(t),\ \mathbf{U}(t)]\geq \mathbf{0} \tag{3}$$

and terminal constraints, D:

$$\mathbf{D}[\mathbf{X}(t_f),\ \dot{\mathbf{X}}(t_f),\ \ddot{\mathbf{X}}(t_f),\ t_f]=\mathbf{0} \tag{4}$$

It is assumed that (i) an optimal trajectory exists and is unique, (ii) $\mathbf{X}^*(t)$ and $\dot{\mathbf{X}}^*(t)$ of the optimal trajectory are continuous, and (iii) $\ddot{\mathbf{X}}^*(t)$ is piecewise differentiable. As mentioned in the Introduction, the necessary conditions for optimality can be derived by variational methods, such as Pontryagin's minimum principle, leading to a 2PBVP which is usually difficult to solve.

Proposed Approach

The optimal control problem specified above can be converted into a nonlinear programming problem by directly representing each generalized coordinate by a Fourier series. In particular, by assuming the optimal profile of the ith generalized coordinate, $X_i^*(t)$, to be continuous in the interval $[0,\ t_f]$, its Fourier series, $X_i'(t)$, will converge to $X_i^*(t)$ in $(0,t_f)$, i.e.,

$$X_i^*(t)=X_i'(t)=a_{i0}'+\sum_{m=1}^{\infty} a_{im}'\cos\frac{2m\pi t}{t_f}+\sum_{m=1}^{\infty} b_{im}'\sin\frac{2m\pi t}{t_f} \tag{5}$$

This approach, however, has the following disadvantages:

1. Convergence is guaranteed only in $(0,t_f)$ unless $X_i^*(t)$ has identical boundary values (Tolstov, 1962, p. 20). To satisfy the requirements on arbitrary boundary conditions (e.g., the specified initial conditions), convergence should extend from $(0,t_f)$ to $[0,t_f]$.

2. Although $X_i'(t)$ converges to $X_i^*(t)$, there is no guarantee that the derivative of $X_i'(t)$ will converge to the derivative of $X_i^*(t)$. Convergence on the derivatives is necessary since the first and second derivatives of the generalized coordinates appear explicitly in the governing differential equations of the dynamic system.

3. The rate of convergence of the Fourier series depends on the optimal solution, $X_i^*(t)$. This rate can be quite slow.

One method to overcome the above difficulties is to append to the series a linear function of time such that

$$X_i^*(t) = [X_i^*(t_f) - X_i(0)] \frac{t}{t_f} + a_{i0}'' + \sum_{m=1}^{\infty} a_{im}'' \cos \frac{2m\pi t}{t_f} + \sum_{m=1}^{\infty} b_{im}'' \sin \frac{2m\pi t}{t_f} \quad (6)$$

which can be rewritten as

$$Y_i(t) = a_{i0}'' + \sum_{m=1}^{\infty} a_{im}'' \cos \frac{2m\pi t}{t_f} + \sum_{m=1}^{\infty} b_{im}'' \sin \frac{2m\pi t}{t_f} \quad (7)$$

where

$$Y_i(t) = X_i^*(t) - [X_i^*(t_f) - X_i(0)] \frac{t}{t_f} \quad (8)$$

is a linear function of time with the property

$$Y_i(0) = Y_i(t_f) \quad (9)$$

This property of identical boundary values implies the following:

1. The convergence interval of the Fourier series of $Y_i(t)$ extends from $(0,t_f)$ to $[0,t_f]$ (Tolstov, 1962, p. 20).

2. The first derivative (calculated by term-by-term differentiation) of the Fourier series of $Y_i(t)$ converges to $\dot{Y}_i(t)$ in the interval of $(0,t_f)$ (Tolstov, 1962, p. 133).

3. The rate of convergence is more rapid since $Y_i(t)$ can be viewed as a function with period t_f (Tolstov, 1962, p. 144).

It should be noted that equation (9) only guarantees that the derivative of the Fourier series of $Y_i(t)$ converges to $\dot{Y}_i(t)$ in $(0,t_f)$. This convergence interval can extended to $[0,t_f]$ if $\dot{Y}_i(t)$ has the same value at the endpoints (see Tolstov, 1962, p. 133). That is, higher order convergence on the boundaries can be achieved by equating the boundary values of higher order derivatives. In summary, by adding a suitable linear function of time to $X_i^*(t)$, a new periodic function, $Y_i(t)$, can be generated which has equal values at the boundaries of $[0,t_f]$. Consequently, convergence on the boundaries and term-by-term differentiation can be guaranteed, and the rate of convergence of the Fourier coefficients can be improved.

Similarly, by adding a suitable polynomial of time to $X_i^*(t)$ it is possible to make the function as well as several of its derivatives have equal values at the endpoints of the interval. By doing this, the order of convergence on the boundaries, the order of the term-by-term differentiation, as well as the rate of convergence can be increased. For example, if

$$Y_i(0) = Y_i(t_f) \quad (10)$$

$$\dot{Y}_i(0) = \dot{Y}_i(t_f) \quad (11)$$

$$\ddot{Y}_i(0) = \ddot{Y}_i(t_f) \quad (12)$$

then it can be shown that over the interval $[0,t_f]$, the first and second derivatives of the Fourier series of $Y_i(t)$ will converge to $\dot{Y}_i(t)$ and $\ddot{Y}_i(t)$, respectively. The rate of convergence of the Fourier coefficients of $Y_i(t)$ also becomes three orders faster than the rate of convergence of the Fourier coefficients of $X_i^*(t)$ (Tolstov, 1962, p. 130).

Equations (10)–(12) can be satisfied by appending a third order polynomial (without the constant term) to $X_i^*(t)$. However, in the approach proposed here each generalized coordinate is represented by the sum of a fifth order polynomial and a finite-term Fourier-type series. A fifth order polynomial is employed since a finite-term Fourier-type series introduces a discrepancy between the true optimal trajectory and its Fourier-type approximation. This discrepancy, in general, violates the initial condition requirements on X_i and $\dot{X}_i$ specified in equation (1). By raising the order of the polynomial from three to five, the five coefficients of the polynomial can be adjusted to satisfy equations (10)–(12) as well as the constraints imposed by the initial conditions of the generalized coordinate X_i and its derivative $\dot{X}_i$. (There are only five coefficients since the constant term is not included in the polynomial, but rather the Fourier-type series.)

Thus, the basic idea of the proposed approach is to approximate each generalized coordinate by the sum of a fifth-order polynomial and a finite-term Fourier-type series. For example, the ith generalized coordinate, $X_i(t)$, is represented by

$$X_i(t) = P_i(t) + \lambda_i(t) \quad (13)$$

where

$$P_i(t) = \sum_{j=0}^{5} p_{ij} t^j \quad (14)$$

and

$$\lambda_i(t) = \sum_{m=1}^{M} a_{im} \cos \frac{2m\pi t}{t_f} + \sum_{m=1}^{M} b_{im} \sin \frac{2m\pi t}{t_f} \quad (15)$$

where M is the number of terms included in the Fourier-type series. Here, the constant term of the Fourier-type series has been included in the fifth-order auxiliary polynomial, $P_i(t)$.

The boundary condition requirements can be written:

$$X_i(0) = P_i(0) + \lambda_i(0) \quad (16)$$

$$X_i(t_f) = P_i(t_f) + \lambda_i(t_f) \quad (17)$$

$$\dot{X}_i(0) = \dot{P}_i(0) + \dot{\lambda}_i(0) \quad (18)$$

$$\dot{X}_i(t_f) = \dot{P}_i(t_f) + \dot{\lambda}_i(t_f) \quad (19)$$

$$\ddot{X}_i(0) = \ddot{P}_i(0) + \ddot{\lambda}_i(0) \quad (20)$$

$$\ddot{X}_i(t_f) = \ddot{P}_i(t_f) + \ddot{\lambda}_i(t_f) \quad (21)$$

These equations can be used to determine the coefficients of the auxiliary polynomial in terms of the coefficients of the Fourier-type series and the boundary values of X_i, $\dot{X}_i$, and $\ddot{X}_i$. Solving the above boundary condition equations gives the following closed-form expressions of the six coefficients:

$$p_{i0} = X_{i0} - \sum_{m=1}^{M} a_{im} \quad (22)$$

$$p_{i1} = \dot{X}_{i0} - \frac{2\pi}{t_f} \sum_{m=1}^{M} m b_{im} \quad (23)$$

$$p_{i2} = \frac{1}{2} \ddot{X}_{i0} + \frac{2\pi^2}{t_f^2} \sum_{m=1}^{M} m^2 a_{im} \quad (24)$$

$$p_{i3} = [10(X_{if} - X_{i0}) + 20\pi \sum_{m=1}^{M} m b_{im} - 4\pi^2 \sum_{m=1}^{M} m^2 a_{im}] t_f^{-3} - (6\dot{X}_{i0} + 4\dot{X}_{if}) t_f^{-2} - \left(\frac{3}{2} \ddot{X}_{i0} - \frac{1}{2} \ddot{X}_{if}\right) t_f^{-1} \quad (25)$$

$$p_{i4} = [15(X_{i0} - X_{if}) - 30\pi \sum_{m=1}^{M} m b_{im} + 2\pi^2 \sum_{m=1}^{M} m^2 a_{im}] t_f^{-4} + (8\dot{X}_{i0} + 7\dot{X}_{if}) t_f^{-3} + \left(\frac{3}{2} \ddot{X}_{i0} - \ddot{X}_{if}\right) t_f^{-2} \quad (26)$$

$$p_{i5} = [6(X_{if} - X_{i0}) + 4\pi \sum_{m=1}^{M} mb_{im}]t_f^{-5}$$

$$- (3\dot{X}_{i0} + 3\dot{X}_{if})t_f^{-4} - \frac{1}{2}(\ddot{X}_{i0} - \ddot{X}_{if})t_f^{-3} \quad (27)$$

where $X_{i0} = X_i(0)$, $X_{if} = X_i(t_f)$, and similarly for the corresponding time derivatives. The advantage of such a formulation is that the given initial conditions, X_{i0} and $\dot{X}_{i0}$, can be embedded naturally in the Fourier-based approximation. This feature also applies for problems with fixed (i.e., known) terminal configuration variables and rates.

Characteristics of Proposed Approach

Simultaneous Adjustment of Free Variables. In the implementation, the free boundary values, the Fourier-type coefficients, a_{im}, b_{im}, and the terminal time, t_f (if it is not fixed) are adjusted simultaneously by a nonlinear programming method such that the performance index is minimized without violating the constraints. Note that the terminal time is treated identically as other free variables. In contrast, variational methods require the formulation of an additional necessary condition for optimal terminal time. This further complicates variationally based numerical algorithms.

Convergence. The approach is a near optimal method since only finite terms of Fourier-type series are used to simulate the trajectory. However, when the number of terms, M, approaches infinity, the near optimal performance index converges to a minimum value, i.e., the value of the optimal performance index. In general, this also implies that the near optimal trajectory converges to the optimal trajectory; however, for the special case of bang-bang control, the near optimal trajectory does not converge to the optimal solution at the time(s) of control variable switching. Convergence is guaranteed at all other times. (Example 2 explores this characteristic.)

Appropriate Approximation Functions. Although full expansion Fourier-type functions have been suggested, half sine, half cosine and other eigenfunctions can be used with the appropriate auxiliary polynomial to approximate each generalized coordinate time history as long as the near optimal trajectory converges to the optimal trajectory when the number of terms of the eigenfunctions approaches infinity. Any orthogonal function which satisfies the requirements on convergence at the boundaries and term-by-term differentiation can be used as an approximating function.

Order of Auxiliary Polynomial. If $\ddot{X}_i(t)$ does not appear explicitly in equation (1), the order of the corresponding auxiliary polynomial, $P_i(t)$ can be reduced from five to three. In this case, $U_i(t)$ and the performance index are not functions of $\ddot{X}_i(t)$, and there is no necessity to achieve convergence in the "acceleration" profile $\ddot{X}_i(t)$.

Inverse Dynamic Method. Since the derivatives of the generalized coordinates are obtained by direct analytical differention of equation (13), the system equations (1) are treated as algebraic equations in evaluating the control variables. The computational scheme of the proposed approach is therefore an inverse dynamic method. As a result, no integration of differential equations (such as state and costate equations) is required. The computational cost is thus significantly reduced. (More will be discussed about this later).

Generalization of Proposed Approach

It was assumed above that the number of control variables is equal to the number of generalized coordinates of the system. There are two possible exceptions to this assumption, as follows:

1. *The number of control variables is greater than the number of generalized coordinates.* Systems with this characteristic are called "redundant" systems. To apply the near optimal method to redundant systems, the time history of each redundant control variable is represented by the sum of a linear function and a finite term Fourier-type series (such as equation (6)). The coefficients are then determined together with the other unknowns to minimize the given performance index.

2. *The number of control variables is less than the number of generalized coordinates.* Systems of this type usually can be partitioned into a subsystem:

$$\mathbf{F}_\alpha[\mathbf{X}(t), \dot{\mathbf{X}}(t), \ddot{X}(t), t] = \mathbf{U}(t) \quad (28a)$$

driven by a vector of control variables $\mathbf{U}(t)$ of dimension L equal to the number of control variables, and a complementary subsystem

$$\mathbf{F}_\beta[\mathbf{X}(t), \dot{\mathbf{X}}(t), \ddot{\mathbf{X}}(t), t] = \mathbf{0} \quad (28b)$$

that is free of control variables. From equations (28*a,b*) a set of L generalized coordinates is chosen such that the remaining generalized coordinate variables (i.e., N-L of these variables, where N is the total number of generalized coordinates) can be calculated from equation (28*b*) (e.g., by integration). Each of the L generalized coordinates can be approximated by a Fourier-type function (13), and the control variables can be computed from equation (28*a*). Note that the computational cost of solving equation (28*a*) may depend upon the selection of the L variables to be approximated. Furthermore, the requirements on terminal conditions still must be satisfied by equations (28*a,b*).

Nonlinear Programming Problem

The performance index of equation (2) can be approximated quite simply using the trapezoidal rule by

$$J \approx E[\mathbf{X}(t_f), \dot{\mathbf{X}}(t_f), t_f] + \sum_{k=1}^{K} G[\mathbf{X}(k\Delta t), \dot{\mathbf{X}}(k\Delta t), \mathbf{U}(k\Delta t), k\Delta t]\Delta t \quad (29)$$

where K is the number of steps or intervals, and $\Delta t = t_f/K$. If each generalized coordinate is represented by the Fourier-type approximation of equation (13), then equation (29) can be rewritten as

$$J \approx \Theta(\mathbf{Z}) \quad (30)$$

where Θ reprelsents a scalar function and $\mathbf{Z}$ consists of the coefficients of the Fourier-type series for all generalized coordinates, free boundary conditions for all generalized coordinates, and the terminal time (if free). Note that time does not appear explicitly in equation (30), and the problem has been converted into a nonlinear programming problem of the following general form:

Minimize $\Theta(\mathbf{Z})$ subject to the constraints:

$$\sigma[\mathbf{X}(k\Delta t), \dot{\mathbf{X}}(k\Delta t), \mathbf{U}(k\Delta t)] \geq 0 \text{ for } k = 1, \ldots, K \quad (31)$$

Constraints (31) represent constraints (3). The terminal constraints (4) have been included in the formulation of the auxiliary polynomial.

The constrained minimization problem is solved using algorithms for the unconstrained problem with inclusion of penalty functions on the violation of constraints (31). Since the equations (30) are often nonlinear and complicated, an analytical expression of the gradient of the performance index is usually not available (or is difficult to obtain). For this reason only nongradient optimization methods are considered. In the simulation studies of the following section, two nongradient methods have been implemented on an IBM PC/XT (with 8087 coprocessor) to solve the nonlinear programming problem. These are the Powell (1964) and the Simplex (Nelder and Mead, 1965) methods. Results obtained from both methods have been found to be quite close. For a more complete discussion of nonlinear programming and optimization, the reader is referred to (Beveridge and Schechter, 1970; Fox, 1971; Avriel, 1976; Siddal, 1982; Reklaitis, et al., 1983).

Like variational-based numerical algorithms for optimal control, nonlinear programming algorithms only guarantee determination of a local minimum. The identification of the global minimum usually requires the trial-and-error introduction of different initial guesses. (Another approach is to employ algorithms which incorporate local minimum avoidance mechanisms, such as recently developed by (Cornan et al., 1987).) In the simulation studies reported below, the initial guess of the nonlinear programming problem was selected by setting the Fourier-type coefficients to zero, i.e., the initial guess was represented by the auxiliary polynomial whose coefficients were determined assuming that unknown boundary values of X_i, $\dot{X}_i$, and $\ddot{X}_i$ were set to zero. Using this procedure, a near optimal performance index was identified for each of the example problems studied.

Simulation Results

Example 1: This example is adapted from (Kirk, 1970, pp. 198–202) and is divided into three subproblems. The dynamic system is a linear, second order system described by:

$$\ddot{X}(t) + X(t) = U(t) \tag{32}$$

with initial conditions

$$X(0) = 0, \ \dot{X}(0) = 0 \tag{33}$$

In (Kirk, 1970) equation (32) is written as a system of two first order state equation in terms of $X_1(t) = X(t)$ and $X_2(t) = \dot{X}(t)$. The system is to be controlled such that its control effort, defined by various performance indices, is minimized. The admissible states and control variables are not bounded.

The three subproblems (identified as Cases) and their closed-form optimal solutions, derived in (Kirk, 1970), are listed below.

Case A. Peformance Index:

$$J = \frac{1}{2} \int_0^2 U^2(t)dt \tag{34}$$

Terminal Constraints:

$$X_1(2) = 5, \ X_2(2) = 2 \tag{35}$$

Solution:

$$X_1^*(t) = 7.289t - 6.103 + 6.696e^{-t} - 0.593e^t \tag{36}$$

$$X_2^*(t) = 7.289 - 6.696e^{-t} - 0.593\ e^t \tag{37}$$

Case B. Performance Index:

$$J = \frac{1}{2}\ [X_1(2) - 5]^2 + \frac{1}{2}\ [X_2(2) - 2]^2 + \frac{1}{2} \int_0^2 U^2(t)dt \tag{38}$$

Terminal Constraints:

None.

Solution:

$$X_1^*(t) = 2.697t - 2.422 + 2.560e^{-t} - 0.137e^t \tag{39}$$

$$X_2^*(t) = 2.697 - 2.560e^{-t} - 0.137e^t \tag{40}$$

Case C. Performance Index:

$$J = \frac{1}{2} \int_0^2 U^2(t)dt \tag{41}$$

Terminal Constraints:

$$X_1(2) + 5X_2(2) = 15 \tag{42}$$

Solution:

$$X_1^*(t) = 0.894t - 1.379 + 1.136e^{-t} + 0.242e^t \tag{43}$$

$$X_2^*(t) = 0.894 - 1.136e^{-t} + 0.242e^t \tag{44}$$

In Case A, the free boundary conditions are $\ddot{X}_1(0)$ and $\ddot{X}_1(2)$, since the initial and terminal state variables are specified. In Case B, the free boundary conditions are $\ddot{X}_1(0)$, $X_1(2)$, $\dot{X}_1(2)$, and $\ddot{X}_1(2)$, since there is no constraint on the terminal state. In Case C, the free boundary conditions are $\ddot{X}_1(0)$, $\ddot{X}_1(2)$, and $\dot{X}_1(2)$ or $X_1(2)$. Here $X_1(2)$ is selected as being free while $\dot{X}_1(2)$ is calculated according to $\dot{X}_1(2) = [15 - X_1(2)]/5$ to assure satisfaction of the terminal constraint.

The three cases of this problem were solved using the Fourier-based approach with $X_1(t)$ represented by equation (13) with a one-term ($M = 1$) Fourier-type series. The performance index was evaluated by means of Simpson's composite integral formula with a step size of 1/30 (consistent unit of time). The optimal values of the free variables and the performance index are listed in Table 1. From the table it can be seen that for all three cases the Fourier-based optimal solutions achieve high accuracy, although only a one-term series was used.

The time history of $X_1(t)$ and $U(t)$ of the optimal solution (from the closed-form expressions) and the near optimal solution (from the Fourier-based method) are plotted in Figs. 1–3. The auxiliary polynomial is also plotted for the three cases for comparison with X_1. The figures show that the curves of the optimal and near optimal solutions are essentially coincident. The figures also show that the auxiliary polynomial approximates the optimal (and near optimal) solutions. This phenomenon is typically exhibited for smooth optimal trajectories, which occur for most optimal control problems. There are, however, situations in which the Fourier-type series represents a significant deviation from the auxiliary polynomial.

Example 2: This example considers a bang-bang control problem, adapted from (Leondes and Wu, 1971), of the linear second order system:

$$\dot{X}_1(t) = X_2(t) \tag{45}$$

$$\dot{X}_2(t) = X_2(t) - X_1(t) + U(t) \tag{46}$$

with initial conditions

$$X_1(0) = 0.231, \ X_2(0) = 1.126 \tag{47}$$

The control is bounded according to the following constraint:

$$-0.8 \le U(t) \le 0.8 \tag{48}$$

The performance index is:

$$J = \frac{1}{2} \int_0^5 [X_1^2(t) + X_2^2(t)]dt \tag{49}$$

From simulation studies, values of the performance index J

Table 1 Fourier-based optimal values for example 1 and comparison to optimal performance index

	Case A	Case B	Case C
a	−1.9564395E−03	−1.6720597E−03	4.3833045E−04
b	1.4421728E−03	5.7709717E−04	1.4015869E−03
$\ddot{X}_1(0)$	6.1025137E+00	2.4175164E+00	1.3739699E+00
$X_1(2)$	5.0000000E+00	2.3031333E+00	2.3530569E+00
$\dot{X}_1(2)$	2.0000000E+00	1.3351959E+00	2.5293861E+00
$\ddot{X}_1(2)$	−3.4798053E+00	−6.7735566E−01	1.9403533E+00
J	1.675072E+01	7.408455E+00	6.708092E+00
Optimal J	1.674543E+01	7.405776E+00	6.702766E+00
% Error	<0.04%	<0.04%	<0.08%

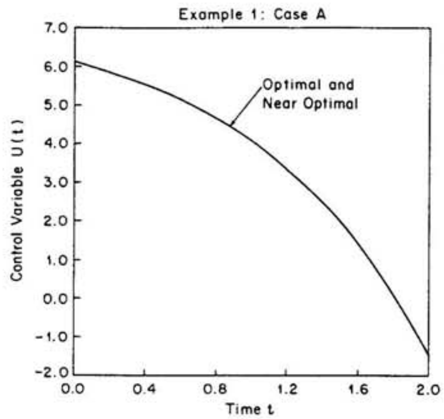

Fig. 1(a)

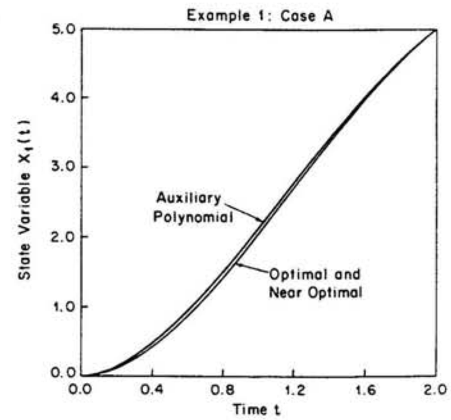

Fig. 1(b)

Fig. 1 History of (*a*) control variable *U*, and (*b*) state variable X_1 and auxiliary polynomial for case A of example 1

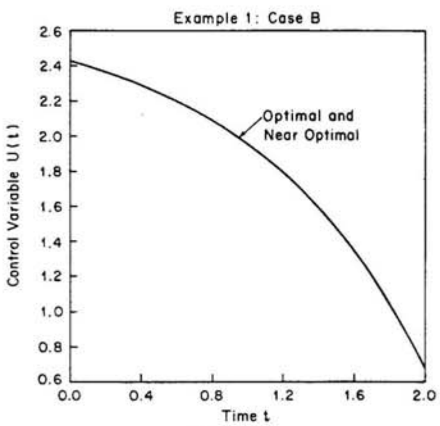

Fig. 2(a)

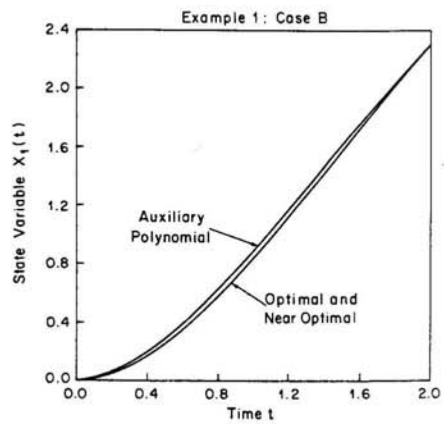

Fig. 2(b)

Fig. 2 History of (*a*) control variable *U*, and (*b*) state variable X_1 and auxiliary polynomial for case B of example 1

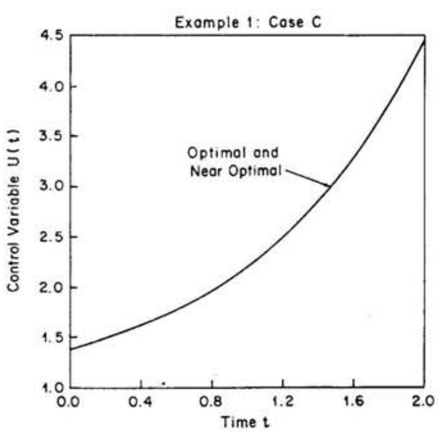

Fig. 3(a)

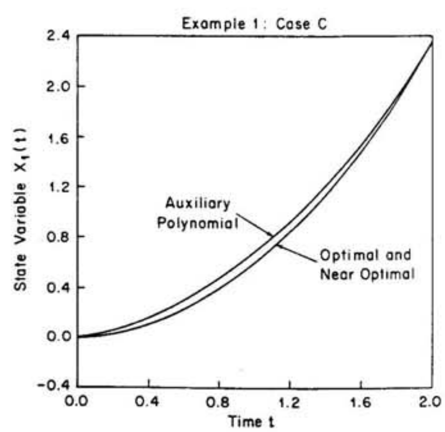

Fig. 3(b)

Fig. 3 History of (*a*) control variable *U*, and (*b*) state variable X_1 and auxiliary polynomial for case C of example 1

were determined to be: (i) 10.34 for the near optimal solution with a two term Fourier-type series, (ii) 9.25 for a three term series, and (iii) 9.04 for a four term series. This compares with a value of J of 5.86 for the optimal solution (Leondes and Wu, 1971). The Fourier-based optimal performance index converges slowly to its minimum value.

The time responses of the control and generalized coordinate variables are plotted in Figs. 4(*a-c*). The control variable time response shows that the Fourier-based method predicts approximately the switching point (i.e., the finite jump). In addition, the solution tends to converge to the optimal solution although the speed of convergence is slow. This property of slow convergence is similar to the "Gibbs' phenomenon" (Wylie, 1975, pp. 247-249), which occurs when developing the Fourier series for a square wave function. The state variable response demonstrates a similar phenomenon. Although the state variable X_1 of the bang-bang solution appears smooth at the finite jump, its corresponding derivative, X_2, is not smooth.

The simulation results suggest that the Fourier-based optimal control approach is not suitable for solution of bang-bang control problems (or for problems with similar rapidly changing response characteristics.) However, bang-bang control is a mathematical idealization which can only be approached in practice due to the finite jump. In contrast, the Fourier-based optimal solution provides a continuous and therefore physically implementable control law. The accuracy of the method can be improved by increasing the number of terms of the series, although this increases the computational cost.

Example 3: This example is adapted from (Kirk, 1970, pp. 338–341). The state equations of the system are highly nonlinear and coupled:

$$\dot{X}_1(t) = -2[X_1(t)+0.25] + [X_2(t)+0.5]\exp\left[\frac{25X_1(t)}{X_1(t)+2}\right] - [X_1(t)+0.25]U(t) \quad (50)$$

$$\dot{X}_2(t) = 0.5 - X_2(t) - [X_2(t)+0.5]\exp\left[\frac{25X_1(t)}{X_1(t)+2}\right] \quad (51)$$

(Here, $\exp[\zeta] = e^{\zeta}$.) The equations are derived from a model of a continuous stirred-tank chemical reactor described in (Lapidus and Luus, 1967). The initial conditions are:

$$X_1(0) = 0.05,\ X_2(0) = 0.00 \quad (52)$$

The performance index is

$$J = \int_0^{0.78} [X_1^2(t) + X_2^2(t) + 0.1U^2(t)]dt \quad (53)$$

In this problem there are two generalized coordinates (X_1 and X_2) and one control variable (U), and the state equations can be represented by equations (28*a*,*b*). To solve this problem, $X_1(t)$ is first approximated by the sum of a fifth order polynomial and a one-term Fourier-type series. The time history of $X_2(t)$ is obtained from the second state equation by direct numerical integration using a fourth order Runge-Kutta method. Finally, the control variable $U(t)$ is calculated from the first state equation.

In (Kirk, 1970) this problem was solved using three different numerical methods: steepest-descent, variation of extremals, and quasilinearization. Results of these methods as well as the Fourier-based method are listed in Table 2. The value of the performance index from the Fourier-based optimal control algorithm falls within the values from the other methods.

The time history of the control variable and the time history of the two state variables are plotted in Figs. 5(*a*) and 5(*b*), respectively, for the different methods. (The results of the method of quasilinearization are not plotted.) From the figures it is observed that the solution of the near optimal method approximates well the solutions of the method of steepest descent and of the method of variation of extremals. In summary, the results show that the Fourier-based approach handles successfully this coupled, nonlinear problem.

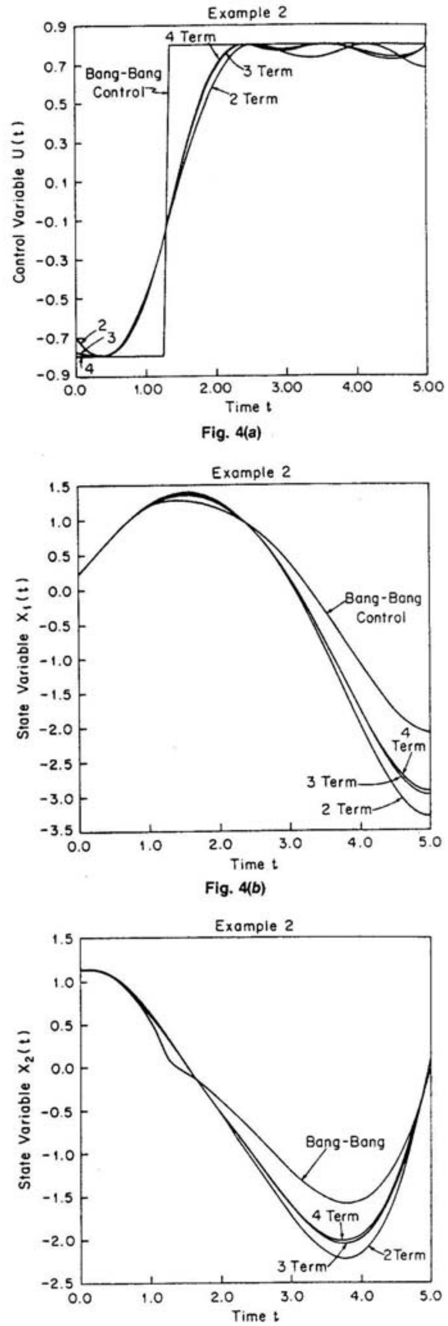

Fig. 4 History of (*a*) control variable *U*, (*b*) state variable X_1, and (*c*) state variable X_2 for example 2

Example 4: This example is adapted from (Kirk, 1970, pp.

Table 2 Values of performance index from different methods for example 3

Method	J
Steepest descent	0.02668
Variation of extremals	0.02660
Quasilinearization	0.02660
Fourier-based	0.02662

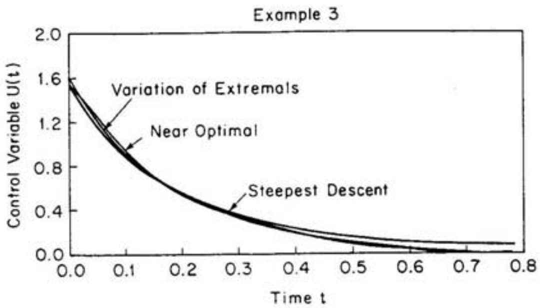

Fig. 5(a)

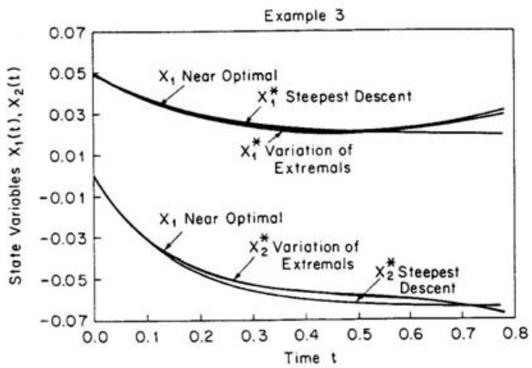

Fig. 5(b)

Fig. 5 History of (a) control variable U, and (b) state variables X_1 and X_2 for example 3

Table 3 Comparison of results for example 4

Method	J	$X_1(0.78)$	$X_2(0.78)$
Gradient projection	0.0022	$-6.167E-06$	$-0.631E-06$
Fourier-based method	0.0021	$<1.0E-09$	$<1.0E-09$

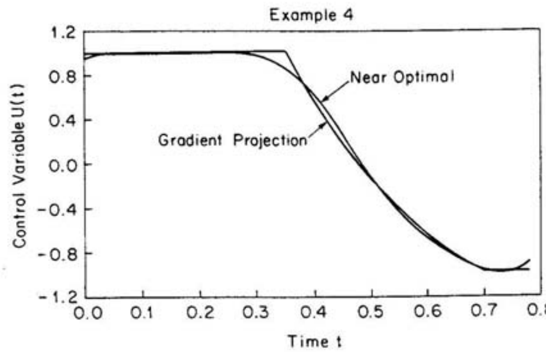

Fig. 6(a)

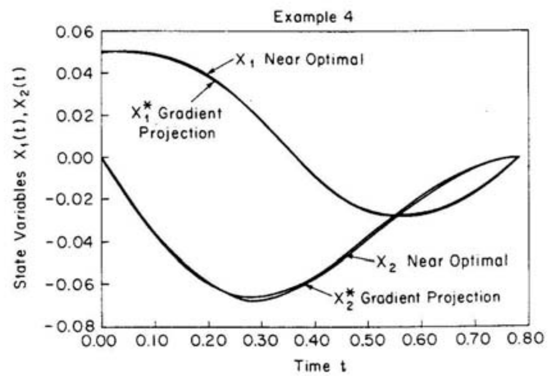

Fig. 6(b)

Fig. 6 History of (a) control variable U, and (b) state variables X_1 and X_2 for example 4

405–407). The state equations and initial conditions are identical to those of Example 3. Here, however, the performance index is independent of control U and is changed to:

$$J=\int_0^{0.78}[X_1^2(t)+X_2^2(t)]dt \tag{54}$$

In addition, two sets of constraints are imposed. The first set bounds the control as follows:

$$-1.0\leq U(t)\leq 1.0 \text{ for } t\in[0\quad 0.78] \tag{55}$$

The second set of constraints fixes the terminal state as follows:

$$X_1(0.78)=0.0,\ X_2(0.78)=0.0 \tag{56}$$

As in Example 3, this example addresses a system with more generalized coordinates than control variables. Here, the problem is further complicated due to the constraints fixing the terminal state. To solve this problem, X_2 is approximated by the sum of a fifth order polynomial and a two term Fourier-type series. $\dot{X}_2(0.78)$ is known from the second state equation with $X_1(0.78) = X_2(0.78) = 0.0$, whereas $\dot{X}_1(0.78)$ is free.

The control variable is a function of $X_1(t)$, $\dot{X}_1(t)$, and $\dot{X}_2(t)$. $X_1(t)$ can be obtained explicitly in terms of the approximation for $X_2(t)$, and $\dot{X}_2(t)$ directly from the second state equation. $\dot{X}_1(t)$ can then be found by direct differentiation. Thus, the control variable can be determined, its constraints can be checked, and the performance index can be evaluated.

In (Kirk, 1970) this problem was solved using the gradient projection method. The values of the performance index as well as the terminal values of the generalized coordinates are given in Table 3. The values from the Fourier-based optimal control method agree closely to the results in (Kirk, 1970).

The time history of the control variable and the history of the generalized coordinates are plotted in Figs. 6(a) and 6(b), respectively. The control variable determined from the gradient projection method shows two nondifferentiable points at the boundaries of the saturation regions of the control variable. In contrast, the near optimal control is differentiable over the entire interval, although it approaches the regions of saturation associated with the optimal solution. Despite the differences in the control histories, the generalized coordinate histories for the two methods are close.

This example demonstrates the application of the Fourier-based method for the solution of a constrained, nonlinear, optimal control problem. The satisfaction of the final boundary conditions as well as the close match of the performance index to that of an independent method (the gradient projection method) verify the method's effectiveness.

Example 5: This example demonstrates the performance of the method in solving a variable terminal time, i.e., t_f free, optimal control problem. Here, a two degree of freedom structural system is considered. The system model is described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad (57)$$

with initial conditions

$$\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (58)$$

The performance index is

$$J = \frac{1}{2} \mathbf{X}^T(t_f) \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \mathbf{X}(t_f)$$

$$+ \int_0^{t_f} [\mathbf{X}^T \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{X} + \mathbf{U}^T \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{U}] dt \quad (59)$$

Unlike the previous examples which have fixed terminal times, the terminal time of this problem is constrained by $0 < t_f \leq 1$.

The value of the performance index as a function of terminal time, t_f, is plotted in Fig. 7(*a*). Here, for every value of t_f the corresponding optimal control problem was solved by a linear quadratic solver based on exponential evaluation of the Hamiltonian matrix (Speyer, 1986). The optimal t_f was found to be 0.18320 with a performance index value of 32.93968. For comparison, using the Fourier-based approach (with two terms), the optimal t_f was found to be 0.18301 with the associated near optimal performance index value of 32.93987. The X_1 state trajectories obtained by both approaches, as shown in Fig. 7(*b*), are very close. Similar agreement is found for the X_2 and control trajectories. In summary, by adjusting simultaneously the free boundary values, the coefficients of the Fourier-type series, and the terminal time, t_f, the proposed approach successfully predicts the near optimal solution of a non-fixed terminal time problem.

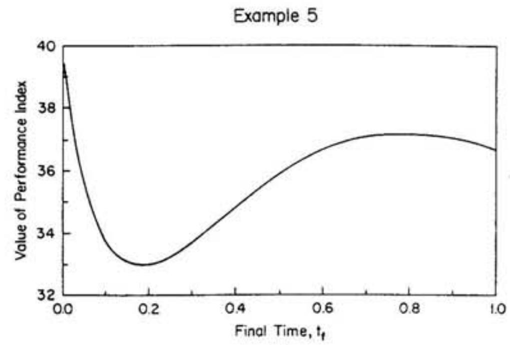

Fig. 7(*a*)

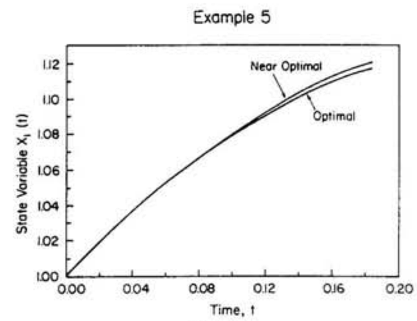

Fig. 7(*b*)

Fig. 7 (*a*) Value of performance index as function of terminal time, and (*b*) history of state variable X_1 for example 5

Discussion

Comparison of Fourier-Based and Gradient-Based Methods. In the Fourier-based method, the derivatives of the generalized coordinates are obtained by analytical differentiation of equation (13), and thus the system equations (1) are treated as algebraic equations in evaluating the control variables. As such, the Fourier-based method is an *inverse dynamic* approach. It finds the optimal solution by adjusting the trajectory itself and the control variables become the "outputs."

In contrast, gradient-based methods are *direct dynamic* approaches. In these methods, optimality is achieved by adjusting the control variables (which are viewed as the "unknowns") based on the results of integrating forward the state equations and integrating backward the costate equations, i.e., solving a 2PBVP. The control variables are modified such that a specified performance index is minimized. Although different 2PBVPs can be posed to achieve optimality, the integration of a system of differential equations (usually evolved from state and costate equations) is indispensable.

An important consequence of the distinction between the inverse and direct dynamic approaches relates to numerical error. The direct dynamic approach involves integration of differential equations in which computational errors often tend to propagate. In fact, it is the accumulation of errors that frequently leads to failure in solving optimal control problems for high dimension systems. In an inverse dynamic approach, the control variables are obtained by straightforward algebra. The only significant computational error is due to the numerical integration of the performance index. Usually, this error can be controlled and estimated easily.

Comparison of Fourier-Based and Dynamic Programming Methods. The primary difference between dynamic programming and the Fourier-based method is the way each approximates the time history of the trajectory. Dynamic programming typically divides the time history of the generalized coordinates into a finite number of intervals. Optimality is achieved by finding the optimal values of the generalized coordinates at each time interval. The Fourier-based approach, on the other hand, approximates the time history of each generalized coordinate by a single function. Optimality is found by adjusting a (typically) much smaller number of parameters (i.e., the free coefficients of the functions), and the computer memory requirement is therefore greatly reduced.

A further advantage of the Fourier-based optimal control approach is its ease in handling problems with free terminal time. In contrast, there does not seem to be an efficient implementation of dynamic programming that addresses the free terminal time problem. A possible advantage of dynamic programming is its ability to guarantee identification of a global minimum. However, nonlinear programming identified what appears to be a global minimum without difficulty for each example problem of the Fourier-based optimal control approach.

Quality of Fourier-Based Solution. A direct way to verify the quality of the Fourier-based optimal control law is to check if it satisfies the necessary conditions for optimality which are derived by variational techniques. In practice, this verification can be done by substituting the near optimal solution into an appropriate, standard, optimal control algorithm and determining if the termination criterion of the selected algorithm can be satisfied.

For instance, the Fourier-based results of Case B of Example 1 were substituted into the steepest descent method (Kirk, 1970, pp. 335-337). Whereas "true" optimality is represented by

$$\|\partial H/\partial \mathbf{U}\| = 0 \tag{60}$$

where H is the Hamiltonian, the norm for the near optimal solution was less than 10^{-6}, suggesting that the optimal solution had been closely approximated.

An alternative empirical approach to verify the optimality is to append another term (i.e., increase M) of the Fourier-type series to the previous solution and re-execute the Fourier-based optimal control algorithm. Additional terms can be added, on a term by term basis, until the value of the performance index converges, indicating that the optimal solution has been reached. This empirical scheme, however, may not be appropriate for bang-bang control problems since under such circumstances the near optimal solution typically has a slow convergence rate.

Summary

This paper develops a Fourier-based method for generating near optimal trajectories of dynamic systems represented by deterministic, lumped-parameter models. The method is conceptually simple, computationally robust, and applicable to a broad class of physical problems. Simulation studies demonstrate the effectiveness of the proposed approach for handling linear, unconstrained problems as well as nonlinear, constrained problems, while sidestepping many of the numerical difficulties typically encountered in implementing optimal control theory. The results show that for all problems, except bang-bang control problems, the near optimal trajectories achieve optimality accurately and with high speed of convergence.

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