Nagurka, M.L. and Yen, V., "Near Optimal Trajectory Planning of Linearly Constrained Structural Systems via State Parameterization," Proceedings of the 31st Structures, Structural Dynamics and Materials Conference, Paper AIAA-90-1091-CP, Long Beach, CA, April 2-4, 1990, pp. 295-301.

AIAA-90-1091-CP

Near Optimal Trajectory Planning of Linearly Constrained Structural Systems via State Parameterization

M. Nagurka*   V. Yen**

Carnegie Mellon University
Pittsburgh, Pennsylvania

Abstract

Based on the idea of state parameterization, this paper develops a Fourier-based approach for solving unconstrained and linearly constrained linear quadratic (LQ) optimal control problems involving structural systems. It is shown that these problems can be converted into quadratic programming problems that can readily be solved. In particular, the necessary condition of optimality for unconstrained LQ problems is obtained as a system of linear algebraic equations. An example problem demonstrates the approach for handling LQ problems with state constraints.

Introduction

The optimal control of linear, lumped parameter, dynamic systems is the subject of much theoretical and practical interest, and is well covered in many textbooks[1-4]. Typically, the necessary condition of optimality is formulated as a two-point boundary-value problem (TPBVP) using variational methods. Except in some special cases, the solution of this TPBVP is usually difficult, and in some cases not practical, to obtain.

In contrast to variational methods, trajectory parameterization approaches[5-10] offer an alternative strategy for solving optimal control problems. In general, these techniques approximate the control and/or state vectors by functions with unknown coefficients, thereby converting an optimal control problem into a mathematical programming (MP) problem. A near optimal solution can then be obtained via various well developed optimization algorithms.

A direct application of trajectory parameterization is to represent the control variables by a sequence of eigenfunctions with unknown weighting coefficients. Then, a linear or nonlinear programming algorithm can be used to determine the values of the coefficients (*i.e.*, control parameters) such that a performance index is minimized. A difficulty with control parameterization occurs in determining the functional relationship between the state variables and control parameters. The process of determining this relationship often requires numerical integration of the state equations, which can be computationally intensive and sensitive to numerical errors.

Approaches based on state parameterization have been described[11-15]. In these approaches, state parameters are adjusted by an MP algorithm in order to minimize a performance index. For example, it has been proposed[14] to represent the time history of each generalized coordinate of a dynamic system by an auxiliary polynomial and a finite-term Fourier-type series. The free variables (*i.e.*, state parameters), such as the (free) coefficients of the polynomial and the Fourier-type series, are adjusted by a MP method. A challenge of state parameterization involves the problem of trajectory inadmissibility, *i.e.*, due to constraints on the control structure an arbitrary representation of the state trajectory may not be achievable.

Finally, combined state and control parameterization approaches have been suggested. For example, in one approach[16] both the state and control variables are expanded in Chebyshev series and an algorithm is provided for approximating the system dynamics, boundary conditions and performance index. Although an advantage of this approach is that it can handle linear as well as nonlinear problems, a drawback is the tedious analytical formulation required for different optimal control problems. In addition, since both the state and control variables are parameterized, the number of free variables is typically higher than the number employed in either state or control parameterization approaches.

Of the different trajectory parameterization approaches, state parameterization offers two major advantages. First, boundary condition requirements on the state variables can be handled directly. Second, if the trajectory inadmissibility problem can be overcome, the state equations can be used as algebraic equations. As a result, the process of determining the functional relationship between the state and control vectors is easier to implement in state parameterization than in control parameterization.

This research is part of a broader effort toward the development of a computational tool for solving optimal control problems via state parameterization. As part of this effort, this paper presents a specialized version of a Fourier-based state parameterization approach[13] for determining the optimal trajectories of linear structural systems with quadratic performance indices and linear constraints. Such linearly constrained linear quadratic (LQ) structural problems are converted into quadratic programming (QP) problems which can be solved by well developed routines.

*Associate Professor, Department of Mechanical Engineering.
**Post-Doctoral Assistant

Problem Statement

The behavior of a controlled linear structure is governed by the equation of motion:

$$\mathbf{M}(t)\ddot{\mathbf{x}}(t) + \mathbf{C}(t)\dot{\mathbf{x}}(t) + \mathbf{K}(t)\mathbf{x}(t) = \mathbf{B}(t)\mathbf{u}(t)\,,\ \ \mathbf{x}(0) = \mathbf{x}_o,\ \dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_o \qquad (1)$$

where $\mathbf{x}$ is an $N \times 1$ configuration vector (*i.e.*, a column vector of N configuration variables), $\mathbf{u}$ is a $J \times 1$ control vector, $\mathbf{M}$ is an $N \times N$ positive definite mass matrix, $\mathbf{C}$ is an $N \times N$ positive semidefinite structural damping matrix, $\mathbf{K}$ is an $N \times N$ positive semidefinite stiffness matrix, and $\mathbf{B}$ is an $N \times J$ control influence matrix. (In this paper vectors are denoted by boldface lower case letters and matrices are represented by boldface upper case letters.) It is assumed that J is less than or equal to N, *i.e.*, the number of control variables is less than or equal to the number of configuration variables. The derivation below considers the case $J = N$ with $\mathbf{B}$ nonsingular implying that every configuration variable can be "actively" controlled. This assumption will be relaxed later.

The design goal is to find the control $\mathbf{u}(t)$ in time interval $[0,T]$ such that the quadratic performance index, L,

$$L = \mathbf{z}^T(T)\,\mathbf{H}\mathbf{z}(T) + \int_0^T (\mathbf{z}^T\mathbf{Q}\mathbf{z} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,dt \qquad (2)$$

is minimized while satisfying the equation of motion (1) and the following system constraints

$$\mathbf{E}_1(t)\mathbf{x}(t) + \mathbf{E}_2(t)\dot{\mathbf{x}}(t) + \mathbf{E}_3(t)\mathbf{u}(t) \leq \mathbf{e}(t) \qquad (3)$$

In equation (2) superscript T represents transpose, whereas T (italics) represents the terminal time. Vector $\mathbf{z}$ is a state vector defined as:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} \qquad (4)$$

It is assumed that $\mathbf{H}$ and $\mathbf{Q}$ are real, nonnegative-definite symmetric matrices and $\mathbf{R}$ is a positive-definite symmetric matrix. In addition, it is assumed that $\mathbf{Q}$ can be partitioned as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_a & \frac{1}{2}\mathbf{Q}_c \\ \frac{1}{2}\mathbf{Q}_c & \mathbf{Q}_b \end{bmatrix} \qquad (5)$$

Thus, the performance index can be rewritten as

$$L = L_1 + L_2 \qquad (6)$$

where

$$L_1 = \mathbf{z}^T(T)\,\mathbf{H}\mathbf{z}(T) \qquad (7a)$$

$$L_2 = \int_0^T (\mathbf{x}^T\mathbf{Q}_a\mathbf{x} + \dot{\mathbf{x}}^T\mathbf{Q}_b\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{Q}_c\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,dt \qquad (7b)$$

L_1 is the cost associated with the terminal configuration and its rate and L_2 is the cost associated with the trajectory. It is assumed that the configuration and control vectors are not bounded, the terminal time T is fixed and the terminal configuration $\mathbf{x}(T)$ is free or fixed.

Fourier-Based Parameterization

The basic idea of the proposed parameterization approach is to approximate each of the N configuration variables $x_n(t)$ by the sum of a fifth-order auxiliary polynominal and a K term Fourier-type series, *i.e.*, for $n = 1,..., N$,

$$x_n(t) = \sum_{k=0}^{5} d_{nk}\, t^k + \sum_{k=1}^{K} \left(a_{nk} \cos\frac{2k\pi t}{T} + b_{nk} \sin\frac{2k\pi t}{T} \right) \qquad (8)$$

The inclusion of the auxiliary polynomial in this representation improves the speed of convergence and differentiability in comparison to a standard Fourier series expansion.[14]

The six coefficients of the auxiliary polynominal can be written as functions of the boundary values of the configuration variables and the coefficients of the Fourier series. In particular, introducing $\nu_k = 2k\pi$, then

$$d_{no} = x_{no} - \sum_{k=1}^{K} a_{nk}\,, \quad d_{n1} = T\dot{x}_{no} - \sum_{k=1}^{K} \nu_k b_{nk} \qquad (9),(10)$$

$$d_{n2} = \frac{1}{2}\left(T^2\ddot{x}_{no} + \sum_{k=1}^{K} \nu_k^2 a_{nk} \right) \qquad (11)$$

$$d_{n3} = 10(-x_{no} + x_{nT}) + (-6\dot{x}_{no} - 4\dot{x}_{nT})T + \left(-\frac{3}{2}\ddot{x}_{no} + \frac{1}{2}\ddot{x}_{nT}\right)T^2 - \sum_{k=1}^{K} \nu_k^2 a_{nk} + 10\sum_{k=1}^{K} \nu_k b_{nk} \qquad (12)$$

$$d_{n4} = 15(x_{no} - x_{nT}) + (8\dot{x}_{no} + 7\dot{x}_{nT})T + \left(\frac{3}{2}\ddot{x}_{no} - \ddot{x}_{nT}\right)T^2 + \frac{1}{2}\sum_{k=1}^{K} \nu_k^2 a_{nk} - 15\sum_{k=1}^{K} \nu_k b_{nk} \qquad (13)$$

$$d_{n5} = 6(-x_{no} + x_{nT}) + 3(-\dot{x}_{no} - \dot{x}_{nT})T + 0.5(-\ddot{x}_{no} + \ddot{x}_{nT})T^2 + 6\sum_{k=1}^{K} \nu_k b_{nk} \qquad (14)$$

where x_{no}, $\dot{x}_{no}$, $\ddot{x}_{no}$, x_{nT}, $\dot{x}_{nT}$, and $\ddot{x}_{nT}$ are the values of the configuration variable x_n and its first and second derivatives at the boundaries of the time segment $[0,T]$, *i.e.*,

$$x_{no} = x_n(0)\,,\ \dot{x}_{no} = \dot{x}_n(0)\,,\ \ddot{x}_{no} = \ddot{x}_n(0) \qquad (15a\text{-}c)$$

$$x_{nT} = x_n(T)\,,\ \dot{x}_{nT} = \dot{x}_n(T)\,,\ \ddot{x}_{nT} = \ddot{x}_n(T) \qquad (15d\text{-}f)$$

Following substitution of equations (9)-(14) into (8), equation (8) can be rearranged in the form

$$x_n(t) = \rho_1 x_{no} + \rho_2\dot{x}_{no} + \rho_3\ddot{x}_{no} + \rho_4 x_{nT} + \rho_5\dot{x}_{nT} + \rho_6\ddot{x}_{nT} + \sum_{k=1}^{K} (\alpha_k a_{nk} + \beta_k b_{nk}) \qquad (16)$$

where

$$\rho_1 = (1 - 10\tau^3 + 15\tau^4 - 6\tau^5) \qquad (17)$$

$$\rho_2 = T(1 - 6\tau^3 + 8\tau^4 - 3\tau^5) \qquad (18)$$

$$\rho_3 = T^2(0.5\tau^2 - 1.5\tau^3 + 1.5\tau^4 - 0.5\tau^5) \qquad (19)$$

$$\rho_4 = (10\tau^3 - 15\tau^4 + 6\tau^5) \qquad (20)$$

$$\rho_5 = (10\tau^3 - 15\tau^4 + 6\tau^5) \qquad (21)$$

$$\rho_6 = \frac{1}{2}T^2(\tau^3 - 2\tau^4 + \tau^5) \qquad (22)$$

$$\alpha_k = -1 + \nu_k^2(0.5\tau^2 - \tau^3 + 0.5\tau^4) + \cos(\nu_k\tau) \qquad (23)$$

$$\beta_k = \nu_k(-\tau + 10\tau^3 - 15\tau^4 + 6\tau^5) + \sin(\nu_k\tau) \qquad (24)$$

with

$$\tau = \frac{t}{T} \qquad (25)$$

The parameters defined in equations (17)-(24) are configuration variable independent and, since the terminal time T is known, are functions of time t only.

Equation (16) can be written more compactly as

$$x_n(t) = \rho^T(t)\, \mathbf{y}_n \tag{26}$$

where

$$\rho^T(t) = [\, \rho_1 \quad \rho_2 \quad \rho_3 \quad \rho_4 \quad \rho_5 \quad \rho_6 \quad \alpha_1 \cdots \alpha_K \quad \beta_1 \cdots \beta_K] \tag{27}$$

and

$$\mathbf{y}_n = [\, x_{no} \quad \dot{x}_{no} \quad \ddot{x}_{no} \quad x_{nT} \quad \dot{x}_{nT} \quad \ddot{x}_{nT} \quad a_{n1} \cdots a_{nK} \quad b_{n1} \cdots b_{nK}]^T$$

$$= [\, y_{n1} \quad y_{n2} \quad \cdots \quad y_{nM}]^T \tag{28}$$

are vectors of dimension $M = 6 + 2K$. The first six elements of $\mathbf{y}_n$ are the values of x_n, $\dot{x}_n$ and $\ddot{x}_n$ at the boundaries of $[0, T]$; the remaining elements are the coefficients of the Fourier-type series. Vector $\mathbf{y}_n$ can be viewed as a time-independent configuration parameter vector for x_n since it characterizes the trajectory of x_n over the time interval $[0, T]$. (Note that the first two elements are known since they are the given initial conditions.)

From equation (26), a vector of the N configuration variables can be written in terms of a full configuration parameter vector $\mathbf{y}$, *i.e.*,

$$\mathbf{x}(t) = \bar{\rho}(t)\mathbf{y} \tag{29}$$

where

$$\bar{\rho} = \begin{bmatrix} \rho^T & & 0 \\ & \rho^T & \\ & & \ddots \\ 0 & & & \rho^T \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} [y_{11} \cdots y_{1M}]^T \\ [y_{21} \cdots y_{2M}]^T \\ \vdots \\ [y_{N1} \cdots y_{NM}]^T \end{bmatrix} \tag{30a,b}$$

Note that $\mathbf{y}$ is a time-independent column vector of dimension NM and $\bar{\rho}$ is a time dependent matrix of dimension $N \times NM$. Similarly, the time derivatives of the configuration vector are

$$\dot{\mathbf{x}}(t) = \bar{\sigma}(t)\mathbf{y}\,, \quad \ddot{\mathbf{x}}(t) = \bar{\gamma}(t)\mathbf{y} \tag{31),(32}$$

where

$$\bar{\sigma}(t) = \dot{\bar{\rho}}(t)\,, \quad \bar{\gamma}(t) = \ddot{\bar{\rho}}(t) \tag{33),(34}$$

Since $\mathbf{x}(t)$, $\dot{\mathbf{x}}(t)$, and $\ddot{\mathbf{x}}(t)$ can be written as functions of $\mathbf{y}$, the control vector $\mathbf{u}(t)$ can be expressed from equation (1) as a function of $\mathbf{y}$:

$$\mathbf{u}(t) = \mathbf{B}^{-1}(t)[\mathbf{M}(t)\ddot{\mathbf{x}}(t) + \mathbf{C}(t)\dot{\mathbf{x}}(t) + \mathbf{K}(t)\mathbf{x}(t)] \tag{35}$$

Substituting equations (29), (31), and (32) into equation (35) gives

$$\mathbf{u}(t) = \mathbf{B}^{-1}(t)[\mathbf{M}(t)\bar{\gamma} + \mathbf{C}(t)\bar{\sigma} + \mathbf{K}(t)\bar{\rho}]\,\mathbf{y} \tag{36}$$

Thus, using the Fourier-based parameterization approach the configuration vector, (the first and second time) derivatives of the configuration vector, and the control vector can be represented as functions of the configuration parameter vector $\mathbf{y}$. It is shown in the following sections that by employing this representation LQ structural problems can be reformulated as QP problems with the configuration variable parameters, y_{nm}, $n = 1,\ldots, N$, $m = 1,\ldots, M$, as variables.

Unconstrained LQ Problems

Conversion Process

The first step in the conversion process from an unconstrained LQ problem to a QP problem is to rewrite the performance index as a function of the configuration parameter vector $\mathbf{y}$. The terminal state part of the performance index L_1 can be written as a function of $\mathbf{y}$ by noting that the terminal state vector is linearly related to $\mathbf{y}$, *i.e.*,

$$\mathbf{z}(T) = \Theta\mathbf{y} \tag{37}$$

where Θ is a $2N \times MN$ transformation matrix with elements 1 and 0, specified according to

$$\theta_{ij} = \begin{cases} 1\,, & j = (i-1)M + 4 \quad \text{for } i = 1,\cdots,N \\ & j = (i-N-1)M + 5 \quad \text{for } i = N+1,\cdots,2N \\ 0\,, & \text{otherwise} \end{cases} \tag{38}$$

By substituting equation (37) into equation (6), the cost L_1 is

$$L_1 = \mathbf{y}^T(\Theta^T\mathbf{H}\Theta)\,\mathbf{y} \tag{39}$$

Similarly, the trajectory part of the performance index L_2 can be written as a function of $\mathbf{y}$, although the process is somewhat more complicated. Substituting equations (31), (32), and (36) into the integrand of equation (7) gives:

$$L_2 = \mathbf{y}^T\Lambda\,\mathbf{y} \tag{40}$$

where

$$\Lambda = \int_0^T (\,\mathbf{F}_1 \otimes \gamma\gamma^T + \mathbf{F}_2 \otimes \sigma\sigma^T + \mathbf{F}_3 \otimes \rho\rho^T + \mathbf{F}_4 \otimes \gamma\sigma^T + \mathbf{F}_5 \otimes \gamma\rho^T + \mathbf{F}_6 \otimes \sigma^T\rho)\,dt \tag{41}$$

with

$$\mathbf{F}_1 = \mathbf{M}^T\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{M}\,, \quad \mathbf{F}_2 = \mathbf{C}^T\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{C} + \mathbf{Q}_b \tag{42),(43}$$

$$\mathbf{F}_3 = \mathbf{K}^T\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{K} + \mathbf{Q}_a\,, \quad \mathbf{F}_4 = 2\mathbf{M}^T\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{C} \tag{44),(45}$$

$$\mathbf{F}_5 = 2\mathbf{M}^T\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{K}\,, \quad \mathbf{F}_6 = 2\mathbf{C}^T\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{K} + \mathbf{Q}_c \tag{46),(47}$$

where superscript -T denotes inverse transpose. Matrices $\mathbf{F}_1$ to $\mathbf{F}_6$ depend on system parameters and performance weighting. (For simplicity, the time-dependent symbol (t) has been omitted in the above equations.) In equation (41), $\otimes$ is the Kronecker product sign.[17]

For problems with time-varying system parameters and/or performance weighting, matrices $\mathbf{F}_1$ to $\mathbf{F}_6$ are functions of time and, in general, the integral of equation (41) must be evaluated numerically. For time-invariant problems, these matrices are constants and can be removed from the integral; the remaining integral parts of Λ can then be evaluated analytically. That is, for time-invariant problems equation (41) can be rewritten as

$$\Lambda = \mathbf{F}_1 \otimes \left[\int_0^T (\gamma\gamma^T)\,dt\right] + \mathbf{F}_2 \otimes \left[\int_0^T (\sigma\sigma^T)\,dt\right] + \mathbf{F}_3 \otimes \left[\int_0^T (\rho\rho^T)\,dt\right]$$
$$+ \mathbf{F}_4 \otimes \left[\int_0^T (\gamma\sigma^T)\,dt\right] + \mathbf{F}_5 \otimes \left[\int_0^T (\gamma\rho^T)\,dt\right] + \mathbf{F}_6 \otimes \left[\int_0^T (\sigma^T\rho)\,dt\right] \tag{48}$$

The integrals in the brackets of equation (48) have been evaluated in closed-form. As a result, the Fourier-based approach is numerically integration-free in handling time-invariant problems.

In summary, by employing equations (39) and (40), the performance index L can be written as a quadratic function of parameter vector $\mathbf{y}$, *i.e.*,

$$L = \mathbf{y}^T \Psi \mathbf{y} \tag{49}$$

where

$$\Psi = \Lambda + \Theta^T \mathbf{H} \Theta \tag{50}$$

The optimization problem can thus be viewed as the search for y_{nm}, $n = 1,..., N$, $m = 1,..., M$, that minimizes the performance index of equation (49) subject to the equality constraints

$$y_{n1} = x_{no}\,,\ y_{n2} = \dot{x}_{no} \quad for\ \ n = 1\ ,..., N \tag{51}$$

representing the initial conditions where x_{no} is the initial value of the n-th configuration variable and $\dot{x}_{no}$ is the initial value of the n-th configuration variable rate.

Solution Procedure

The equality constrained QP problem (outlined above) is solved by converting it into an unconstrained QP problem. To accomplish this goal, a new configuration parameter vector $\mathbf{v}$ is introduced, specified as

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \tag{52}$$

where

$$\mathbf{v}_1^T = [\ \mathbf{a}^T \quad \mathbf{b}^T \quad \ddot{\mathbf{x}}_o^T \quad \ddot{\mathbf{x}}_T^T \quad \dot{\mathbf{x}}_T^T \quad \mathbf{x}_T^T\ ] \tag{53}$$

$$\mathbf{v}_2^T = [\ \mathbf{x}_o^T \quad \dot{\mathbf{x}}_o^T\ ] \tag{54}$$

with

$$\mathbf{x}_o = [\ x_{1o} \quad x_{2o} \cdots x_{No}\ ]^T\ ,\quad \mathbf{x}_T = [\ x_{1T} \quad x_{2T} \cdots x_{NT}\ ]^T \tag{55),(56}$$

$$\dot{\mathbf{x}}_o = [\ \dot{x}_{1o} \quad \dot{x}_{2o} \cdots \dot{x}_{No}\ ]^T\ ,\quad \dot{\mathbf{x}}_T = [\ \dot{x}_{1T} \quad \dot{x}_{2T} \cdots \dot{x}_{NT}\ ]^T \tag{57),(58}$$

$$\ddot{\mathbf{x}}_o = [\ \ddot{x}_{1o} \quad \ddot{x}_{2o} \cdots \ddot{x}_{No}\ ]^T\ ,\quad \ddot{\mathbf{x}}_T = [\ \ddot{x}_{1T} \quad \ddot{x}_{2T} \cdots \ddot{x}_{NT}\ ]^T \tag{59),(60}$$

$$\mathbf{a} = [\ a_{11} \cdots a_{1K} \quad a_{21} \cdots a_{2K} \quad \cdots\cdots \quad a_{N1} \cdots a_{NK}\ ]^T \tag{61}$$

$$\mathbf{b} = [\ b_{11} \cdots b_{1K} \quad b_{21} \cdots b_{2K} \quad \cdots\cdots \quad b_{N1} \cdots b_{NK}\ ]^T \tag{62}$$

Vector $\mathbf{v}_2$ contains the known initial values of the state vector; vector $\mathbf{v}_1$ is the remaining subset of the parameter vector $\mathbf{y}$ (*i.e.*, obtained by excluding $\mathbf{v}_2$ from $\mathbf{y}$).

The two vectors $\mathbf{v}$ and $\mathbf{y}$ are related via a linear transformation

$$\mathbf{y} = \Phi\, \mathbf{v} \tag{63}$$

where Φ is a $2NM$ x $2NM$ matrix with elements 1 and 0. The performance index L of equation (49) can thus be rewritten as a function of $\mathbf{v}$

$$L = \mathbf{v}^T \Omega \mathbf{v} \tag{64}$$

where

$$\Omega = \Phi^T \Psi \Phi \tag{65}$$

By expanding equation (64), the performance index can be expressed as

$$L = [\ \mathbf{v}_1^T \quad \mathbf{v}_2^T\ ] \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \tag{66}$$

or equivalently

$$L = \mathbf{v}_1^T \Omega_{11} \mathbf{v}_1 + \mathbf{v}_1^T (\Omega_{12} + \Omega_{21}^T) \mathbf{v}_2 + \mathbf{v}_2^T \Omega_{22} \mathbf{v}_2 \tag{67}$$

The performance index of equation (67) is a quadratic function of $\mathbf{v}_1$, the unknown part of the configuration parameter vector. For an unconstrained LQ problem, the necessary condition of optimality can be obtained by differentiating the performance index with respect to this unknown configuration parameter vector. This leads to

$$(\Omega_{11} + \Omega_{11}^T)\, \mathbf{v}_1 = -\, (\Omega_{12} + \Omega_{21}^T)\, \mathbf{v}_2 \tag{68}$$

which represents a system of linear algebraic equations from which the unknown vector $\mathbf{v}_1$ can be solved.

If the terminal value of the configuration vector is known, the same solution procedure can be applied. The only modification required is to redefine the unknown vector $\mathbf{v}_1$ as

$$\mathbf{v}_1^T = [\ \mathbf{a}^T \quad \mathbf{b}^T \quad \ddot{\mathbf{x}}_o^T \quad \ddot{\mathbf{x}}_T^T \quad \dot{\mathbf{x}}_T^T\ ] \tag{69}$$

and the known vector $\mathbf{v}_2$ as

$$\mathbf{v}_2^T = [\ \mathbf{x}_T \quad \mathbf{x}_o \quad \dot{\mathbf{x}}_o^T\ ] \tag{70}$$

Similarly, problems with fixed initial and/or terminal configuration variable rates can be handled.

An interesting feature of equation (68) is that the coefficient matrix of $\mathbf{v}_1$ is independent of known boundary values (*i.e.*, $\mathbf{v}_2$). Thus, for the same unconstrained LQ problem with different boundary values, the coefficient matrix remains the same; only the right-hand side constant vector needs to be recomputed.

Linearly Constrained LQ Problems

The Fourier-based approach is also applicable to linearly constrained LQ problems. Here, the system constraints of equation (3) are converted into a system of linear algebraic constraints.

The approach is to substitute equation (35) into the inequality constraints (3) giving

$$\mathbf{S}_1(t)\mathbf{x}(t) + \mathbf{S}_2(t)\dot{\mathbf{x}}(t) + \mathbf{S}_3(t)\ddot{\mathbf{x}}(t) \le \mathbf{e}(t) \tag{71}$$

where

$$\mathbf{S}_1(t) = \mathbf{E}_1(t) + \mathbf{E}_3(t)\mathbf{B}^{-1}(t)\mathbf{K}(t) \tag{72}$$

$$\mathbf{S}_2(t) = \mathbf{E}_2(t) + \mathbf{E}_3(t)\mathbf{B}^{-1}(t)\mathbf{C}(t) \tag{73}$$

$$\mathbf{S}_3(t) = \mathbf{E}_3(t)\mathbf{B}^{-1}(t)\mathbf{M}(t) \tag{74}$$

Using the configuration variable parameterization of equations (29), (31), and (32) in (71) gives

$$\mathbf{G}(t)\mathbf{y} \le \mathbf{e}(t) \tag{75}$$

where

$$\mathbf{G}(t) = \mathbf{S}_1(t)\bar{\rho}(t) + \mathbf{S}_2(t)\bar{\sigma}(t) + \mathbf{S}_3(t)\bar{\gamma}(t) \tag{76}$$

The inequality constraints (75) are functions of time. Consequently, they represent an infinite number of constraints which need to be satisfied along the trajectory. For practicality, these constraints are relaxed to be satisfied only at a finite number of points (usually chosen to be equally spaced) in time. That is, it is required that only a finite number (I) of algebraic inequalities

$$\mathbf{G}(t_i)\mathbf{y} \le \mathbf{e}(t_i) \quad for \quad i = 1,..., I \tag{77}$$

be satisfied. In terms of configuration parameter vector $\mathbf{v}$, the inequalities of (77) can be rewritten using equation (63) as

$$\mathbf{G}^*(t_i)\, \mathbf{v} \le \mathbf{e}(t_i) \quad for \quad i = 1,..., I \tag{78}$$

where

$$\mathbf{G}^{*}(t_i) = \mathbf{G}(t_i)\Phi \tag{79}$$

By decoupling $\mathbf{v}$ into $\mathbf{v}_1$ and $\mathbf{v}_2$, the inequality constraints of (78) can be represented as

$$\begin{bmatrix} \mathbf{G}^{*}_{11}(t_i) & \mathbf{G}^{*}_{12}(t_i) \\ \mathbf{G}^{*}_{21}(t_i) & \mathbf{G}^{*}_{22}(t_i) \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \leq \begin{bmatrix} \mathbf{e}_1(t_i) \\ \mathbf{e}_2(t_i) \end{bmatrix} \quad for \quad i = 1, ..., I \tag{80}$$

Since $\mathbf{v}_2$ is known, the corresponding terms can be moved to the right-hand side of (80) giving

$$\begin{bmatrix} \mathbf{G}^{*}_{11}(t_i) \\ \mathbf{G}^{*}_{21}(t_i) \end{bmatrix} \mathbf{v}_1 \leq \begin{bmatrix} \mathbf{e}_1(t_i) - \mathbf{G}^{*}_{12}(t_i)\mathbf{v}_2 \\ \mathbf{e}_2(t_i) - \mathbf{G}^{*}_{22}(t_i)\mathbf{v}_2 \end{bmatrix} \quad for \quad i = 1, ..., I \tag{81}$$

Thus, the system constraints (3) can be approximated by the linear algebraic inequalities (81).

In summary, by applying Fourier-based parameterization of the configuration variables, a linearly constrained LQ problem can be converted into a QP problem in which the quadratic function of equation (67) is to be minimized without violating the system of linear algebraic inequalities of (81).

General Linear Systems

The approach presented above is applicable to systems with square and invertible control matrices. This section generalizes the Fourier-based approach to the more common case of structural systems which have fewer control variables than configuration variables. The dynamic system of interest is again the linear structural system described by equation (1). In this case, the control influence matrix, $\mathbf{B}$, is an $N \times J$ matrix where the number of configuration variables, N, is greater than the number of control variables, J. It is assumed that the rank of $\mathbf{B}$ is equal to J.

To apply the Fourier-based approach, the equation of motion, equation (1), is first modified to

$$\mathbf{M}(t)\ddot{\mathbf{x}}(t) + \mathbf{C}(t)\dot{\mathbf{x}}(t) + \mathbf{K}(t)\mathbf{x}(t) = \mathbf{B}'(t)\mathbf{u}(t) \tag{82}$$

where

$$\mathbf{B}'(t) = \mathbf{B}'_{N x N} = \begin{bmatrix} \mathbf{I}_{(N-J)x(N-J)} & \mathbf{B}_{NxJ} \\ \mathbf{O}_{Jx(N-J)} & \end{bmatrix} \tag{83}$$

and

$$\mathbf{u}'(t) = \mathbf{u}'_{Nx1} = \begin{bmatrix} \hat{\mathbf{u}}_{(N-J)x1} \\ \mathbf{u}_{Jx1} \end{bmatrix} \tag{84}$$

with the subscripts representing the dimensions of the matrices. By introducing an artificial control vector, $\hat{\mathbf{u}}$, the new control matrix, $\mathbf{B}'$, can be inverted enabling the calculation of the control, $\mathbf{u}'(t)$, for any given trajectory (similar to equation (35)). Note that it can be guaranteed that $\mathbf{B}'$ is invertible if the last J rows of $\mathbf{B}$ are nonsingular. However, if the last J rows are singular, the first $(N-J)$ columns of $\mathbf{B}'$ in equation (83) can always be modified to make it invertible since it has been assumed that $\mathbf{B}$ has rank J.

In order to predict the optimal solution, the performance index is modified to

$$L' = L + r\int_0^T \left[\hat{\mathbf{u}}^T(t)\, \hat{\mathbf{u}}(t) \right] dt \tag{85}$$

where L is the performance index of the original LQ problem and r is a weighting constant chosen to be a large positive number. The integral term associated with r is used to represent the contribution of the artificial control.

The advantage of using artificial control variables is that a non-actively controlled structural system can be converted to an actively controlled structural system to which the Fourier-based approach is applicable. The trade-off is that the resulting solution will not, in a strict mathematical sense, satisfy the trajectory admissibility requirement due to the existence of artificial control variables.[13] However, by penalizing the artificial control vector, the magnitude and influence of the artificial control variables can be made insignificant and the solution of the modified optimal control problem can closely approximate the solution of the original LQ problem.

Example

This example problem, adapted from Evtushenko[10], p. 438, considers a one degree-of-freedom system with a time-varying configuration variable constraint. For the system described by

$$\ddot{x}(t) + \dot{x}(t) = u(t) \quad , \quad x(0) = 0 \; , \; \dot{x}(0) = -1 \tag{86}$$

it is required to find the solution that minimizes the performance index

$$L = \int_0^1 \left[x^2 + \dot{x}^2 + 0.005u^2 \right] dt \tag{87}$$

without violating the constraint

$$\dot{x}(t) \leq e(t) \tag{88}$$

where

$$e(t) = 8(t - 0.5)^2 - 0.5 \tag{89}$$

Previously[10], this problem was solved using a control parameterization approach. Here, the problem was solved using the proposed Fourier-based approach, where the QP solution algorithm of Gill and Murray[18] was implemented to determine the optimal configuration parameter values. The simulations were executed on a SUN-3/60 workstation with the codes written in the "C" language.

The resulting values of the performance index for three to nine term Fourier-type series are summarized in Table I. As shown in this Table, the performance index values decrease as the number of terms of the Fourier-type series increases. In particular, the Fourier-based solutions with series of six and more terms are less than the minimum performance index of 0.17114 obtained by Evtushenko[10]. Furthermore, the differences between the Fourier-based solutions are small (for example, the difference between the eight and nine term Fourier-based solutions is less than 0.09 percent) suggesting that convergence has been achieved.

Table I Summary of Simulation Results using K Term Fourier-Type Series (Evtushenko's Solution is 0.17114)

K	Performance Index
3	0.17480
4	0.17268
5	0.17115
6	0.17069
7	0.17069
8	0.17028
9	0.17013

The response history for $\dot{x}(t)$ obtained with a three term Fourier-type series is plotted in Figure 1. The constraint history and the solution computed by Evtushenko[10] are also plotted in this figure. The Fourier-based solution satisfies the configuration variable constraint and closely approximates the trajectory predicted by Evtushenko. In fact, the Fourier-based solution appears indistinguishable from Evtushenko's solution when the configuration variable constraint is active.

In summary, this example demonstrates the applicability of the Fourier-based approach for handling LQ problems with configuration variable inequality constraints. For the problem studied, the Fourier-based approach yields higher accuracy in predicting the optimal solution in comparison to a previous result. Future example problems will address the computational efficiency of the method and investigate the artificial control variable technique.

Discussion

An advantage of a parameterization approach, such as the Fourier-based approach, is that it characterizes the optimal trajectory (which, in theory, consists of an infinite number of points) by a relatively small number of trajectory parameters. An optimal control problem can thus be converted into an algebraic optimization problem (*i.e.*, a MP problem). In general, the corresponding computations are much less complicated than those involved in standard optimal control solvers.

For unconstrained LQ problems, the performance index, which initially is written as a quadratic functional, is converted into a quadratic function. By differentiating this quadratic function with respect to the free parameters, the necessary condition of optimality is derived as a system of linear algebraic equations which can readily be solved.

For linearly constrained LQ problems, the system constraints are relaxed to be satisfied only at a finite number of points (usually equally-spaced) in time. Consequently, the linear system constraints are replaced by a finite number of linear algebraic inequalities. The optimal control problem is thus converted into a QP problem.

In applying the Fourier-based approach, finite-term Fourier-type series are employed. As a result, the Fourier-based approach can be classified as a near optimal control approach. The accuracy of the Fourier-based approach can be estimated empirically by increasing the number of terms of the Fourier-type series. Additional terms can be added, on a term by term basis, until the value of the performance index converges, indicating that the optimal solution is closely approximated.

Conclusion

This paper shows that by applying a Fourier-based state parameterization approach linearly constrained LQ problems of structural systems can be converted into QP problems. Preliminary simulation results show that the proposed approach is an accurate design tool for determining the optimal solution of such linearly constrained LQ problems.

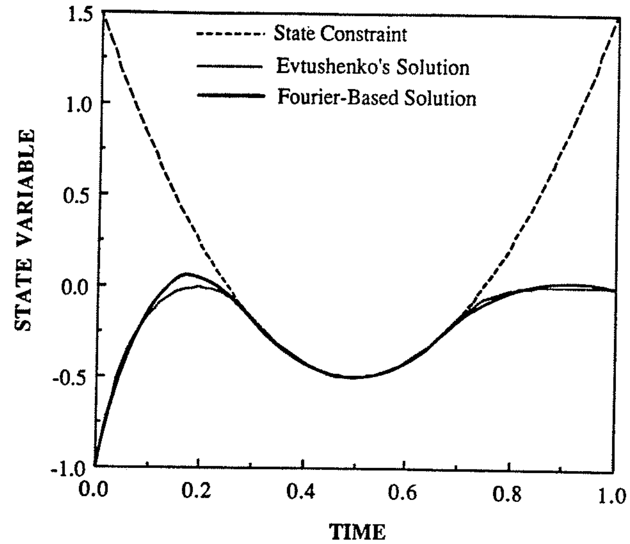

Figure 1. History of Configuration Variable Rate (With Three Term Fourier-Type Series)

References

1. **Athans, M., and Falb, P.L., 1966,** *Optimal Control: An Introduction to the Theory and Its Applications*, McGraw-Hill, Inc., New York, NY.
2. **Kirk, D.E., 1970,** *Optimal Control Theory: An Introduction*, Prentice-Hall, Inc., Englewood Cliffs, NJ.
3. **Sage, A.P., and White, C.C., 1977,** *Optimal Systems Control*, Prentice-Hall, Inc., Englewood Cliffs, NJ.
4. **Lewis, F.L, 1986,** *Optimal Control*, John Wiley & Sons, Inc, New York, NY.
5. **Canon, M.D., Cullum, C.D., and Polak, E., 1970,** *Theory of Optimal Control and Mathematical Programming*, McGraw-Hill, Inc., New York, NY.
6. **Tabak, D., 1970,** "Applications of Mathematical Programming Techniques in Optimal Control: A Survey," *IEEE Transactions on Automatic Control*, Vol. AC-15, pp. 688-690.
7. **Tabak, D., and Kuo, B.C., 1971,** *Optimal Control by Mathematical Programming*, Prentice-Hall, Inc., Englewood Cliffs, NJ.
8. **Luenberger, D.G., 1972,** "Mathematical Programming and Control Theory: Trends of Interplay," *Perspective on Optimization*, Addison-Wesley Publishing Company, Inc., Reading, MA.
9. **Kraft, D., 1980,** "Comparing Mathematical Programming Algorithms Based on Lagrangian Functions for Solving Optimal Control Problems," *Control Applications of Nonlinear Programming*, ed. Rauch, H.E., Pergamon Press, Oxford.
10. **Evtushenko, Y.G., 1985,** *Numerical Optimization Techniques*, Optimization Software, Inc., New York, NY.

11. **Johnson, F.T., 1969,** "Approximating Finite-Thrust Trajectory Optimization," *AIAA Journal*, Vol. 7, June, pp. 993-997.

12. **Nair, G.G, 1978,** "Suboptimal Control of Nonlinear Systems," *Automatica*, Vol. 14, pp. 517-519.

13. **Yen, V. and Nagurka, M.L., 1988,** "Fourier-Based State Parameterization for Linear Quadratic Optimal Control," Technical Paper 88-WA/DSC-7, ASME Winter Annual Meeting, Chicago, IL, November 27 - December 2.

14. **Nagurka, M. L. and Yen, V., 1989,** "Fourier-Based Optimal Control of Nonlinear Dynamic Systems," *ASME Journal of Dynamic Systems, Measurement, and Control*, in press.

15. **Yen, V. and Nagurka, M.L., 1989,** "A Fourier-Based Optimal Control Approach for Structural Systems," *AIAA Journal of Guidance, Control, and Dynamics*, in press.

16. **Vlassenbroeck, J. and Van Dooren, R., 1988,** "A Chebyshev Technique for Solving Nonlinear Optimal Control Problems," *IEEE Transactions on Automatic Control*, Vol. AC-33, pp. 333-340.

17. **Brewer, J.W., 1978,** "Kronecker Products and Matrix Calculus in System Theory," *IEEE Transactions on Circuits and Systems*, Vol. CAS-25, pp. 772-781.

18. **Gill, P.E. and Murray, W., 1977,** "Linearly Constrained Problems including Linear and Quadratic Programming," *The State of the Art in Numerical Analysis*, ed. Jacobs, D., Academic Press.