Gain and Phase Margins of SISO Systems from Modified Root Locus Plots

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This paper promotes graphically based methods for determining the gain margin and phase margin of linear time-invariant single-input, single-output control systems. The gain margin can be found from a graph showing the angle of each closed-loop system eigenvalue in the complex plane as a function of real gain. At any constant real gain, the phase margin can be identified from a graph of the angle of each closed-loop system eigenvalue in the complex plane as a function of gain angle. The proposed methods do not require frequency calculations, and highlight the importance of root sensitivity, with the practical design guideline of not selecting control gains that place eigenvalues near break points.

Classical Presentation of Relative Stability

In the design of control systems one is interested in relative stability as well as absolute stability. Although as Bellman has stated "there is no stability to the definition of stability," a system can be considered absolutely stable if a transient oscillation decays and ultimately vanishes. A system on the border of absolute instability is prone to oscillations that continue for a long time. The overshoot and settling time for step inputs may be excessive, degrading the performance of the system. Furthermore, systems operating near marginal stability may be driven to instability by sensor noise, disturbances, and modeling errors. Thus, a system must be relatively as well as absolutely stable in practice, making robustness a paramount design consideration.

In the time domain, relative stability of a linear time-invariant (LTI), single-input, single-output (SISO) control system is measured by parameters such as the maximum overshoot and the settling time. In the frequency domain, the resonance peak can be used to indicate relative stability. Alternatively, relative stability can be determined by means of the Nyquist plot of the loop transfer function $g(s)$. The proximity of the Nyquist contour ($g(j\omega)$ polar plot) to the critical point, $(-1, j0)$, yields an indication of the closed-loop system's degree of stability.

A typical Nyquist plot or open-loop frequency locus for a minimum phase transfer function $g(s)$ is shown in Fig. 1. (It is assumed that $g(j\omega)$ is a minimum phase transfer function, so that the portion of the Nyquist contour that corresponds to $s=j\omega$, $0 \leq \omega < \infty$, is sufficient for stability analysis.) Application of the Nyquist encirclement test shows that the closed-loop system is absolutely stable. If the loop gain is low, the Nyquist plot of $g(j\omega)$ intersects the negative real axis at a point that is quite far to the right of the critical point. As the gain is increased, the intersection point of the Nyquist contour and the negative real axis moves closer to the critical point, eventually passing through it (corresponding to marginal stability) and then encircling it (indicating instability).

The gain margin is used to quantify the distance between the Nyquist contour intersection of the real axis and the critical point. In general, if the intersection occurs at a distance $|g(j\omega_p)|$ from the origin, then multiplying the gain by a factor $1/|g(j\omega_p)|$ makes the closed-loop system marginally stable. The factor $1/|g(j\omega_p)|$ is the gain margin, and the special frequency ω_p is the phase crossover frequency, i.e., the frequency at the phase-crossover where $\angle g(j\omega_p) = 180^\circ$. In control engineering, it is more common to express this factor in decibels (dB) with a positive gain margin indicating a stable system. The gain margin, GM, in decibels is given by

$$\begin{aligned} GM &= 20 \log\left[1/|g(j\omega_p)|\right] \\ &= -20 \log |g(j\omega_p)| \end{aligned} \tag{1}$$

Thus, the gain margin is the number of decibels by which the magnitude of the open-loop frequency response falls short of unity when the phase angle is 180°.

Since the gain margin is a multiplicative factor on gain, it can be expressed as

$$GM = 20 \log(k^*/k) \tag{2}$$

where k^* is the gain corresponding to marginal stability, i.e., the gain at the crossing of the $j\omega$ axis in the root locus plot. If the root locus does not cross the stability boundary for any gain, the gain margin is infinite.

The phase margin is also a measure of relative stability. It is the angle by which the phase of the open-loop frequency response falls short of −180° when the magnitude is unity. Thus, the phase margin, denoted as PM in Fig. 1, is the additional phase lag required to make the system marginally stable. The phase margin is the phase at the frequency ω_g, the gain crossover frequency, where the magnitude or "gain" of $g(j\omega)$ is unity (0 dB). A positive phase margin indicates a stable system. (In Fig. 1, the phase margin for a minimum phase, open-loop stable system is measured clockwise

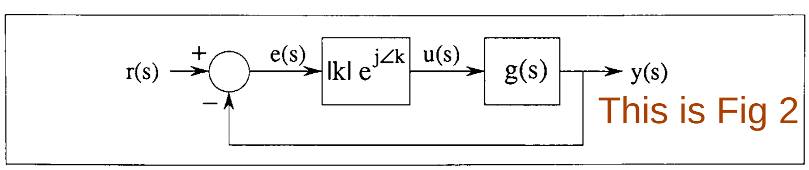

Fig. 1. Nyquist portrait showing conventional definitions of gain margin and phase margin.

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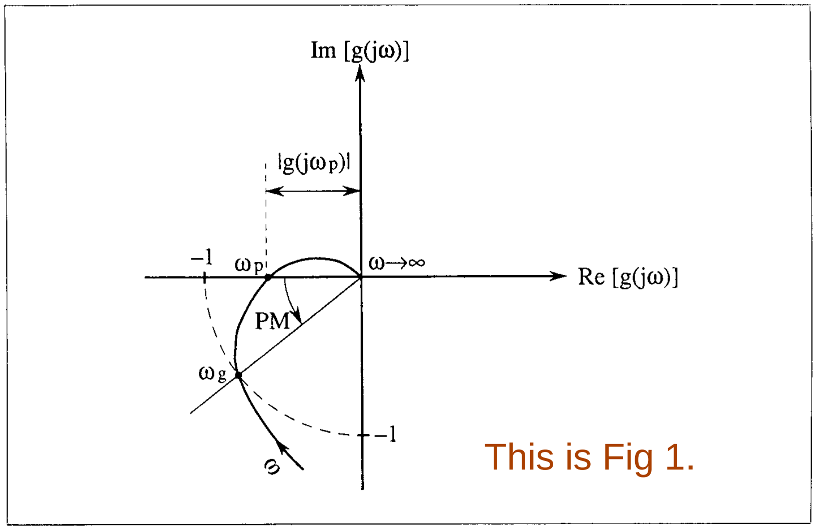

Fig. 2. Feedback block diagram with forward complex gain.

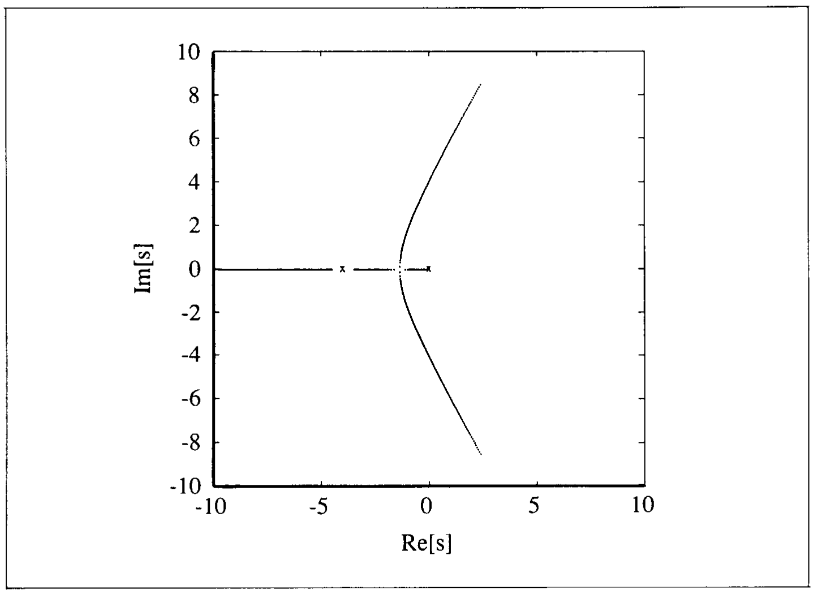

Fig. 3. Root locus plot of example.

from the negative real axis, with a positive phase margin denoting stability.)

The phase and gain margins can be viewed as safety factors in the design specifications. A useful rule-of-thumb generally applicable to control systems is that for adequate closed-loop stability the gain margin should be greater than 6 dB and the phase margin should be between 30° and 60° [1]. (The 6 dB limit corresponds to the quarter amplitude decay response obtained with the gain settings given by the Ziegler-Nichols ultimate-cycle method [2].) Some control engineers offer more restrictive measures, suggesting GM ≥ 8 dB and PM ≥ 40° or even ≥ 50°. These values should be viewed as rough, albeit often useful, working guides. In general, it is not desirable to make the margins too large since this corresponds to low gain systems yielding sluggish designs that may result in large steady-state errors [2].

In this paper, the concepts of gain margin and phase margin are interpreted using an alternate paradigm, namely the feedback block diagram of Fig. 2 where the forward gain is given by $k=|k|\exp(j\angle k)$. The gain margin corresponds to the range $|k|$ can be adjusted, assuming $\angle k=0$, for the closed-loop system to be stable. Similarly, the phase margin corresponds to the range that $\angle k$ can be adjusted for a given $|k|$ such that the closed-loop system is stable. This perspective does not involve the calculation of crossover frequencies, nor does it require Nyquist or Bode plots for illustrating the gain and phase margins.

The gain k in (2) is related to the gain margin under the assumption that the gain is real. (In any physical system the gain is real.) An advantage of employing (2) is that it provides a means to determine the gain margin from relations linking the gain and measures of stability. The most popular graphically-based tool employing gain is the Evans root locus, in which gain is an implicit variable. To exploit the relation of (2) we seek a graphical tool that expresses the information of the root locus in conjunction with the gain continuum (as opposed to discrete tick marks representing gain on the root locus). Since a key assumption in root locus theory is that the gain is real, root locus analysis is limited to calculation of gain margin and not phase margin. By generalizing the forward gain to be a complex quantity with magnitude and phase angle, it is possible to generate a graphical tool to determine phase margin. This paper promotes the use of graphically-based tools for gain margin and phase margin determination.

Gain Margin from Angle-Gain Plot

An alternative graphical representation of the standard root locus plot is to present the magnitude and angle (phase) of the closed-loop system eigenvalues in separate graphs that show the explicit dependency of the forward real gain $k=|k|\exp(j\angle k)=|k|$, where $\angle k=0$. These plots, called the magnitude-gain and angle-gain plots, respectively, have been proposed [3] as a useful pair of plots for control system analysis and design. In fact, they follow from a natural progression of perspectives of the standard root locus in an analogous fashion that the Bode plots are an alternate representation of the Nyquist diagram. By directly exposing the influence

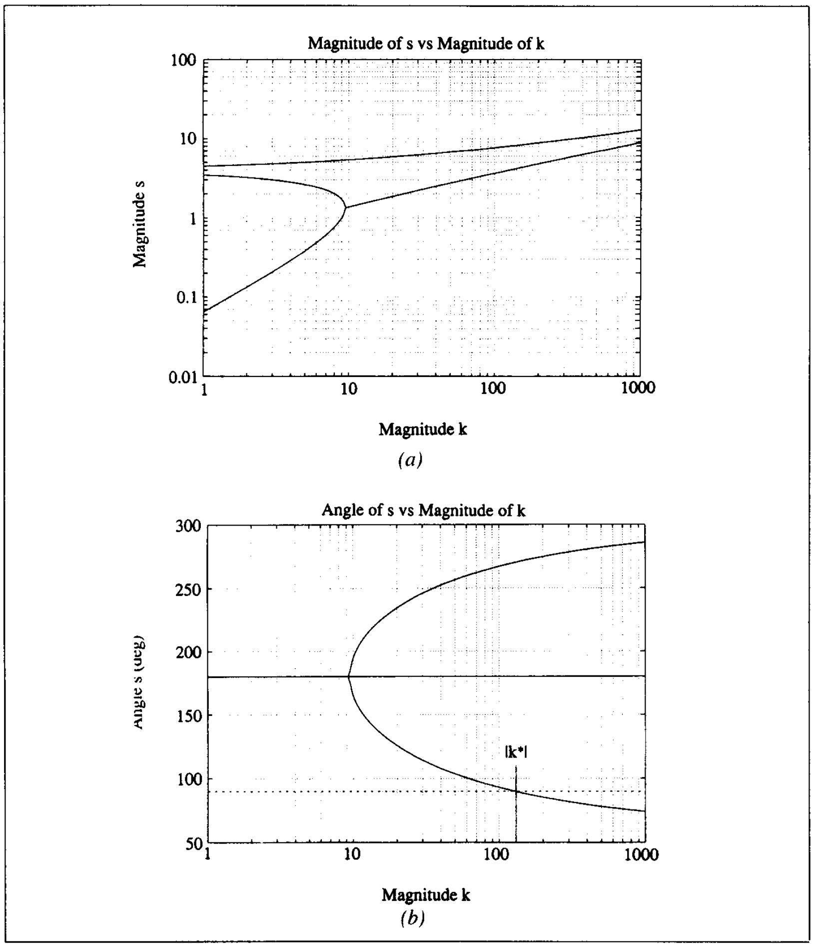

Fig. 4. (a) Magnitude-gain plot, and (b) angle-gain plot of example.

Table I
Results of Gain and Phase Margin Analysis for Example

$\|k\|$	$\|k\|/\|k^*\|$	GM (dB)	PM (°)	ω_p(rad/s)	ω_g (rad/s)
1.00	0.00781	42.14	88.21°	4.0000	0.0625
9.00	0.0703	23.06	74.29°	4.0000	0.5220
9.48	0.0741	22.61	73.49°	4.0000	0.5803
10.0	0.0781	22.14	72.63°	4.0000	0.6108
100	0.781	2.14	7.29°	4.0000	3.5212
128	1.00	0.00	0.00°	4.0000	4.0000
200	1.56	−3.87	−12.06°	4.0000	4.9445
1000	7.81	−17.86	−44.19°	4.0000	9.4672

of gain magnitude on the system eigenvalues, the gain plots enable the designer to select values of gain corresponding to stable behavior that meet desired performance specifications, such as achieving the natural frequency and damping ratio of interest. From the angle-gain plot, the range of gains for which the closed-loop system is stable can be determined by inspection. The magnitude-gain plot is also an important design aid since the slopes of the loci are related to root sensitivity magnitudes [4].

The expression given by (2) is especially well suited for determination of gain margin from the angle-gain plot. In particular, k^*, the gain corresponding to marginal stability, can be determined directly from the angle-gain plot by noting the angle of any eigenvalue that maps to a location on the imaginary axis, i.e., $\angle s = \pm 90°$. Thus, given a design value of magnitude k, (2) can be used to calculate the gain margin. From (2),

$$GM = 20 \log |k^*| - 20 \log |k| \tag{3}$$

indicating that the gain margin is logarithmically proportional to $|k|$.

Gain Margin Example

The open-loop transfer function of this example is given by

$$g(s) = \frac{1}{s(s+4)^2} \tag{4}$$

It is embedded in the closed loop system of Fig. 2 with $k=|k|\exp(j0)$. The root locus is shown in Fig. 3. As the gain is increased the real eigenvalue moves deeper in the left half plane along the real axis whereas the complex conjugate pair of eigenvalues crosses the imaginary axis and enters the right half plane. This behavior is readily observable in the gain plots of Fig. 4(a) and 4(b). By inspection of the angle-gain plot, marginal stability is reached at $k=k^*=128$. From (2), GM = 42.1 − 20 log k dB. At k=100, GM = 2.1 dB indicating that the closed-loop system with unity forward gain can be increased 2.1 dB before the stability margin is reached. At k=200, GM = −3.88 dB, i.e., the gain must be decreased 3.88 dB for stability to be reached. The results for several gain magnitudes are summarized in Table I. Also shown are the phase and gain crossover frequencies calculated from a frequency analysis. In the proposed approach there is no need to compute the phase crossover frequency to determine the gain margin.

Phase Margin from Angle-Angle Plot

By relaxing the constraint that the gain be purely real, it is possible to graphically depict the phase margin in a plot showing the angle of each closed-loop eigenvalue versus the angle of the gain. The perspective of viewing the forward gain as a complex quantity can result in counter-intuitive behavior, e.g., the possibility of generating system eigenvalues that do not occur as complex conjugate pairs.

For a complex forward gain of given magnitude, it is possible to compute and display the root loci showing the closed-loop eigenvalue trajectories in the complex plane as implicit functions of the gain angle. However, sketching rules are not available and there is limited, if any, useful information for the designer. An alternative graphical tool is to depict the angle of each closed-loop system eigenvalue versus the angle of the gain, $\angle k$, for a given real gain $|k|$. We have called this graph the angle-angle plot. The phase margin can be determined by inspection of this graph by identifying the smallest angle of k for which any eigenvalue crosses the instability boundary, i.e., $\pm 90°$ in the complex plane.

An angle-angle plot can be generated for a given gain magnitude. This is analogous to the angle-gain plot which corresponds to a single value of the gain angle, namely $\angle k=0$. A popular example of a zero angle variational magnitude analysis is the Evans root locus plot where $|k|$ only is varied. Thus, it is possible to produce a family of angle-angle plots for different values of $|k|$. Once $|k|$ is chosen, a phase analysis is quite important since gain margin analysis alone does not suffice for determining stability and robustness [5].

In the angle-angle plot, the phase margin is available directly without the use of frequency domain information, i.e., there is no need to compute the gain crossover frequency for the forward loop transmission. In addition to phase margin, the angle-angle plot shows the phase margin sensitivity from the slopes of the curves. Large derivatives indicate that the phase margin is sensitive to angle variations. Sensitivity information is important when considering augmenting the system with other systems such as low pass filters, or

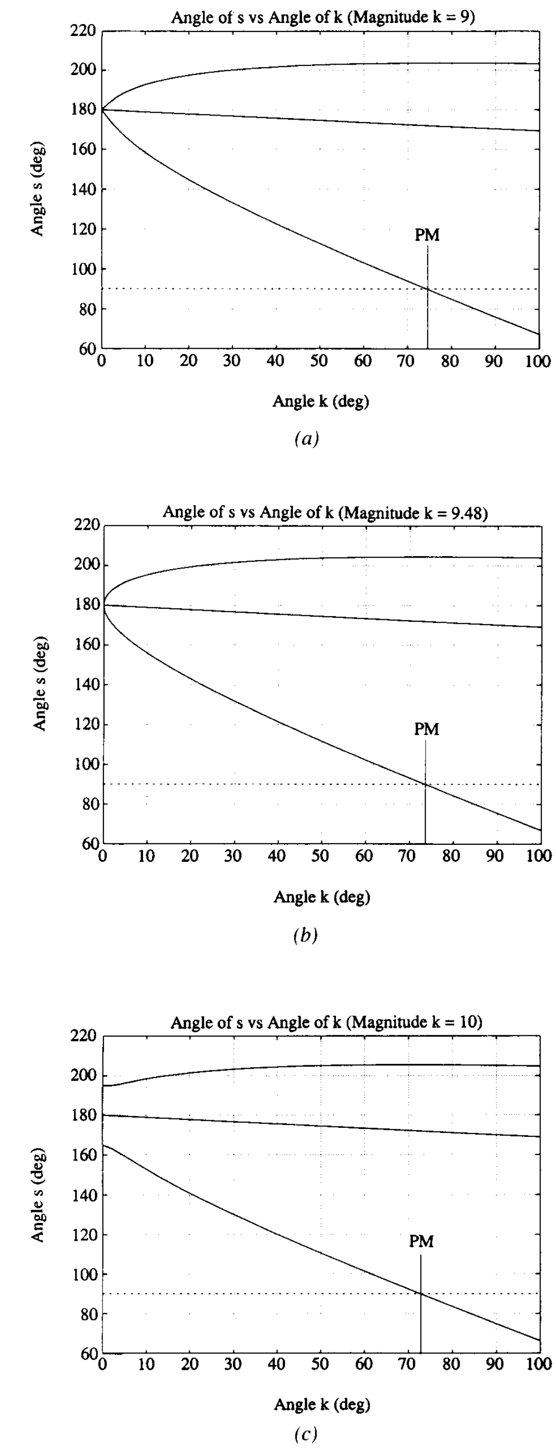

Fig. 5. Angle-angle plots of example for (a) $|k|=9.00$, (b) $|k|=9.48$, and (c) $|k|=10.0$.

when including modeling errors into the control design [6].

Phase Margin Example

We again consider the system given by (4). For the phase margin analysis we are especially interested in $|k| = 9.48$ corresponding to the eigenvalue break-point on the root locus, and gain magnitudes near it (e.g., $|k| = 9.00$ and 10.0). The break-point gain has been isolated to illustrate the corresponding significant changes in the phase margin sensitivity. Table I shows the results, and includes an entry for ω_g, the 0 dB gain crossover frequency obtained via standard Bode plot techniques.

Fig. 5(a)-(c) shows the angle-angle plots for $|k| = 9.00, 9.48$, and 10.0, respectively. From the figure, the phase margin is the angle of k that causes an eigenvalue to cross the 90° line. An interesting attribute is the linear asymptotic behavior of the curves as $\angle k$ increases. A second intriguing feature is the large slope of the curve near $\angle k$=0° in the angle-angle plot for $|k| = 9.48$. This phenomenon is expected since the sensitivity of the system is theoretically infinite at the break-point, and implies that gains placing eigenvalues near the break-points should be avoided when designing control systems. As demonstrated in the example, choosing a gain that is slightly different from the break-point gain can significantly reduce the sensitivity of the phase margin.

Closing

This note presents graphically-based methods for studying relative stability of LTI SISO systems. The angle-gain plot and the angle-angle plot are proposed for finding the gain margin and phase margin, respectively. The angle-gain plot recasts the information of the standard root locus in a form that exposes the explicit functional dependence of forward gain magnitude on the angle of each closed-loop system eigenvalue; the angle-angle plot explicitly relates the forward gain angle to system eigenvalue angles. Furthermore, the sensitivity of the phase margin is available and augments classical design techniques. The proposed methods, recommended for analysis and design of classical control systems, are useful geometric tools that do not require frequency analysis. The proposed framework employs independent gain and phase axes in plots naturally suited for determining gain and phase margin, respectively.

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News

Fraser Steps Down as Editor

Donald Fraser, founding Editor of the *AIAA Journal of Guidance, Control, and Dynamics* announced the end of his tenure in the January-February 1992 issue. Dr. Kyle T. (Terry) Alfried of the General Research Corporation will be the new Editor.

For many years Dr. Fraser was Vice President of the Charles Stark Draper Laboratory in Cambridge MA. At the end of 1990 he moved to Washington DC to become the Deputy Director of Operational Test and Evaluation (Command, Control, Communications, and Intelligence) for the Department of Defense. A year later President Bush nominated Fraser for the post of Deputy Under Secretary of Defense for Acquisition, the number two acquisition official for the entire Department of Defense. In December his appointment was confirmed by the United States Senate, and at that time Fraser decided to step down as Editor — after serving fourteen years since the *Journal* began.

We wish Dr. Fraser well in his new activities.

1991 CDC Proceedings

If you were unable to attend the 1991 Conference on Decision and Control held December 11-13, 1991, in Brighton, England, you can still order the Conference Proceedings. Copies are available at a cost of $100 for IEEE members and $200 for nonmembers. The IEEE catalog number for the Proceedings is 91CH3076-7. To order call (toll free in the U.S. and Canada) 1-800-678-IEEE. From other countries call (908) 981-0060. FAX: (908) 981-9667. Or write: IEEE Customer Service Department, 445 Hoes Lane, P.O. Box 1331, Piscataway, NJ 08855-1331 U.S.A.

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Control Systems Technology Award

Nominations are open for the IEEE Control Systems Society Technology Award, which will be awarded for the fourth time this year. This award is to be given for outstanding contributions to control systems technology, either in design and implementation or in project management. It may be conferred on either an individual or a team. The prize is $1000 and a certificate. Nominations are open to all. The prize is to be awarded at the IEEE Conference on Decision and Control. Deadline for nominations is June 30, 1992. Please send your nominations, together with supporting documents, to the Chair of the Technology Award Committee: Eugene O. King, Aluminum Company of America, PCMT Division–Building B, Alcoa Center, PA 15069. Phone: 412-337-3590. Fax: 412-337-2005.