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Unified H_∞ Approach for a Singularly Perturbed Aircraft Model

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1 Abstract

This paper presents a unified approach to obtain simultaneously the continuous-time and discrete-time state space solutions for the composite state feedback control of singularly perturbed H_∞ systems. First, the scaled frequency domain realizations of dynamic systems are developed and then used to generate the unified H_∞ solutions that approach the H_2 solution as the unified H_∞ norm design parameter is allowed to approach infinity. The realizations also account for the non-orthogonal cost function terms that occur in the unified singularly perturbed slow subsystem. Next, the unified H_2 solutions are provided. Finally, the methodology is then applied to compute the reduced system, composite and full-order controllers gains for the dynamics of an F-8 aircraft model.

2 Introduction

In the analysis and design of continuous-time (CT) and discrete-time (DT) control systems, the theory of singular perturbations and time scales (SPaTS) has been used as a powerful tool for over three decades [1,2]. H_∞-control [3,4], which has been an active topic of research for almost two decades, has received the attention of researchers in the theory of SPaTS [5–7].

The pioneering work by Middleton and Goodwin [8] has aroused interest in *delta*(δ) systems, especially since these systems have better finite wordlength characteristics and fewer conditioning problems in comparison to traditional discrete-time systems. As sampling frequency increases, the differential operator is obtained as a limiting case from the δ-operator (that is, the incremental difference operator), thus establishing a close equivalence with *delta*-systems. This property has lead to the use of the difference operator in several areas of modern multivariable control, including LQR/LQG and H_∞-optimal control.

In this paper, the state space solution for H_∞ composite state feedback control of the unified singularly perturbed systems is developed and then applied to the longitudinal dynamics of an F-8 aircraft model. Two preliminary problems are addressed that lead to the general solution of the H_∞ state feedback control of unified singularly perturbed systems. First, a set of weighting matrices is established such that the H_2-norm of the frequency domain system representation is equivalent to the time domain linear quadratic cost. Then the unified H_∞ solution is shown to approach the H_2 solution as the H_∞-norm design parameter α approaches infinity. Next, a simple transformation of variables is used to eliminate the non-orthogonal terms in the cost function that occur in the unified singularly perturbed problem. Finally, the methodology is applied to solve the near-optimal state feedback control design problem for the longitudinal dynamics of an F-8 aircraft. H_2 and H_∞ solutions are computed for the cases of the full-order control, reduced order control, and composite control. The reduced and composite H_∞ control solutions are then compared with the full-order solution. The unified results given here are generally valid for both the CT case ($\Delta = 0$) and the DT case ($\Delta \neq 0$).

3 The Delta Operator

The approach used here is based on the delta operator [8] defined as

$$\delta = \frac{q-1}{\Delta} \tag{1}$$

where q is the usual forward shift operator and Δ is the sample period. Using the unified notation [8], the state space model obtained is

$$\rho w(\tau) = A_\rho w(\tau) + B_\rho u(\tau) \tag{2}$$
$$y(\tau) = C_\rho w(\tau) \tag{3}$$

where,

$$A_\rho = \left\{ \begin{array}{ll} A & \text{continuous-time} \\ A_\delta & \text{discrete-time} \end{array} \right\}$$
$$\rho = \left\{ \begin{array}{lll} \frac{d}{dt} & \text{continuous-time} & \Delta = 0 \\ \delta & \text{discrete-time} & \Delta \neq 0 \end{array} \right\}$$
$$\tau = \left\{ \begin{array}{ll} t & \text{continuous-time} \\ k & \text{discrete-time} \end{array} \right\}$$

$$S_0^\infty f(\tau)d\tau = \left\{ \begin{array}{lll} \int_0^{t_f} f(t)dt & \text{continuous-time} & \Delta = 0 \\ \Delta \sum_{k=0}^{\infty} f(k\Delta) & \text{discrete-time} & \Delta \neq 0 \end{array} \right\}$$

Note that, for the DT system (2), as $\Delta \to 0, A_\delta \to A$ and $B_\delta \to B$.

4 Preliminaries

The basic unified H_∞ formulation, and the results in system model realizations [8], are applied to the unified singularly perturbed control problem. The transfer function is denoted in terms of the state matrices with the notation

$$G(\gamma) = \left[\begin{array}{c|c} A_\rho & B_\rho \\ \hline C_\rho & D_\rho \end{array}\right] = C_\rho(\gamma I - A_\rho)^{-1}B_\rho + D_\rho \quad (4)$$

A Riccati operator, *Ric*, is introduced as a function that maps the parameter of a Hamiltonian matrix H, into the positive-definite solution of a Riccati equation X. Defining a real Hamiltonian matrix with Q and R symmetric,

$$H = \begin{bmatrix} I & \Delta B_\rho R^{-1}B_\rho' \\ 0 & I + \Delta A_\rho' \end{bmatrix}^{-1} \begin{bmatrix} A_\rho & -B_\rho R^{-1}B_\rho' \\ -Q & -A_\rho{}' \end{bmatrix} \quad (5)$$

The corresponding Riccati equation is

$$0 = A_\rho X + A_\rho' X + \Delta A_\rho' X A_\rho - (B_\rho' X(I + \Delta A_\rho)' (R + \Delta B_\rho' X B_\rho)^{-1}(B_\rho' X(I + \Delta A_\rho)) + Q \quad (6)$$

The following system realization [4] is then used to develop the frequency-domain equivalent unified linear quadratic regulator (ULQR) and the H_∞ state feedback problem:

$$G(\gamma) = \left[\begin{array}{c|cc} A_\rho & B_{1\rho} & B_{2\rho} \\ \hline C_{1\rho} & 0 & D_{12\rho} \\ C_{2\rho} & D_{21\rho} & 0 \end{array}\right]$$

$$\text{where } \bar{x} = \begin{bmatrix} x \\ w \\ u \end{bmatrix} \; \bar{y} = \begin{bmatrix} \rho x \\ z \\ y \end{bmatrix} \quad (7)$$

with assumptions

1. $(A_\rho, B_{1\rho})$ is stabilizable and $(C_{1\rho}, A_\rho)$ detectable
2. $(A_\rho, B_{2\rho})$ is stabilizable and $(C_{2\rho}, A_\rho)$ detectable
3. $D_{12\rho}'[C_{1\rho} \; D_{12\rho}] = [0 \; I]$
4. $\begin{bmatrix} B_{1\rho} \\ D_{21\rho} \end{bmatrix} D_{21\rho}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$

The ULQR problem involves determining the state feedback matrix K_2 that minimizes a time domain cost function:

$$J = \frac{1}{2} S_0^\infty \left[x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau)\right] d\tau \quad (8)$$

with $Q = Q' \geq 0$ and $R > 0$. Next consider,

$$\rho x(\tau) = A_\rho x(\tau) + B_\rho u(\tau) \quad (9)$$

We now let the transfer function matrix from w to z, denoted by T_{zw}, be represented in terms of the lower linear fractional transformation $F_l(G, K)$. Using an appropriately weighted system realization, the equivalent frequency domain ULQR problem can be formulated that determines K_2 to minimize

$$J_2 = \| T_{zw}(\gamma) \|_2 = \| F_l(G, K_2) \|_2 \quad (10)$$

Similarly, the unified H_∞ state feedback control problem involves determining K_∞ that minimizes

$$J_\infty = \| T_{zw}(\gamma) \|_\infty = \| F_l(G, K_\infty) \|_\infty \quad (11)$$

The frequency domain realization for these problems with system equation (9) is given by

$$G(\gamma) = \left[\begin{array}{c|cc} A_\rho & I & B_\rho \\ \hline \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \\ I_{A_\rho} & 0 & 0 \end{array}\right] \quad (12)$$

The above system realization contains appropriate weighting matrices $B_{1\rho}, C_\rho, D_{12\rho}$ and $D_{21\rho}$ such that the transfer function matrix two-norm is equivalent to the time-domain unified linear quadratic cost function. The values for the weighting matrices continue to satisfy assumption 3 in the original problem statement system realizations. In addition, when the appropriate scaling matrices are used in the H_∞ problem, the solution approaches the H_2 solution as the H_∞-norm bound α, approaches infinity. The H_∞ state feedback solution is then

$$u_\infty(\tau) = K_\infty x(\tau)$$

where

$$K_\infty = -(R + \Delta B_\rho' K_\infty B_\rho)^{-1} B_\rho' K_\infty (I + \Delta A_\rho)x$$

$$X_\infty = Ric(H_\infty) \geq 0$$

$$H_\infty = \begin{bmatrix} I & -\Delta(\frac{1}{\alpha^2}B_{1\rho}B_{1\rho}' - B_{2\rho}B_{2\rho}') \\ 0 & I + \Delta A_\rho{}' \end{bmatrix}^{-1} \begin{bmatrix} A_\rho & \frac{1}{\alpha^2}B_{1\rho}B_{1\rho}' - B_{2\rho}B_{2\rho}' \\ -Q & -A_\rho' \end{bmatrix}$$

$$H_\infty = \begin{bmatrix} I & -\Delta(\frac{1}{\alpha^2}I - B_\rho R^{-1}B_\rho') \\ 0 & (I + \Delta A_\rho') \end{bmatrix}^{-1} \begin{bmatrix} A_\rho & \frac{1}{\alpha^2}I - B_\rho R^{-1}B_\rho' \\ -Q & -A_\rho' \end{bmatrix}$$

So as $\alpha \to \infty, H_\infty \to H_2$ as $u_\infty \to u_2$ for the ULQR problem. Also the unified Hamiltonian converge towards the CT Hamiltonian as $\Delta \to 0$ [5].

In assumption 3 the H_∞ formulation effectively places restriction on terms in the cost function. As known in the analysis of singularly perturbed system the slow subsystem involves the non-orthogonal terms in the cost function. In order to eliminate this problem a change of variables or other method is used we consider the analysis with cross products terms.

Consider the following dynamic model and linear quadratic cost function with the cross-product weighting terms.

$$\begin{aligned} &\rho x(\tau) = Ax(\tau) + Bu(\tau) \\ &(Q - SR^{-1}S') = (Q - SR^{-1}S')' \geq 0, R = R' > 0 \quad (13) \\ &J = \frac{1}{2} S_0^\infty \left[x'(\tau)Qx(\tau) + 2x'(\tau)Su(\tau) + u'(\tau)Ru(\tau)\right] d\tau \end{aligned}$$

As suggested in [5], a transformation of variables is used to obtain the solution that minimizes the above cost function. Then we obtain the H_2 state-feedback as:

$$u_2(\tau) = K_2 x(\tau) \quad (14)$$

where

$$K_2 = -\left[(R + \Delta B_\rho' X_2 B_\rho)^{-1}(B_\rho' X_2 (I + \Delta A_\rho) + S')\right] \quad (15)$$

with

$$X_2 = Ric(H_2)$$

$$H_2 = \begin{bmatrix} I & \Delta B_\rho R^{-1}B_\rho' \\ 0 & I + \Delta\tilde{A}_\rho' \end{bmatrix}^{-1} \begin{bmatrix} \tilde{A}_\rho & -B_\rho R^{-1}B_\rho' \\ -\tilde{Q} & -\tilde{A}_\rho' \end{bmatrix}$$

where $\tilde{A}_\rho = A_\rho - B_\rho R^{-1}S'$ and $\tilde{Q} = (Q - SR^{-1}S')$. The scaled frequency-domain realization for the ULQR problem is then given by

$$G(\gamma) = \left[\begin{array}{c|cc} \tilde{A}_\rho & I_{\tilde{A}_\rho} & B_\rho R^{-1/2} \\ \hline \begin{bmatrix} \tilde{Q}^{1/2} \\ 0 \\ I_{\tilde{A}_\rho} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_R \\ 0 \end{bmatrix} \end{array}\right] \quad (16)$$

The above system realization is such that the transfer function matrix two-norm is equivalent to the time domain linear quadratic cost. The realization results in an H_∞ solution that approaches the H_2 solution as the H_∞-norm bound α approaches infinity. The general unified H_∞ state feedback solution is then

$$u_\infty(\tau) = K_\infty x(\tau) \quad (17)$$

where

$$\begin{aligned} K_\infty &= -\Big[(R + \Delta B_\rho' X_\infty B_\rho)^{-1} \\ &\quad (B_\rho' X_\infty (I + \Delta A_\rho) + S')\Big] \end{aligned} \quad (18)$$

with

$$\begin{aligned} X_\infty &= Ric(H_\infty) \\ H_\infty &= \begin{bmatrix} I & -\Delta(\frac{1}{\alpha^2} I - B_\rho R^{-1} B_\rho') \\ 0 & (I + \Delta \tilde{A}_\rho') \end{bmatrix}^{-1} \\ &\quad \begin{bmatrix} \tilde{A}_\rho & \frac{1}{\alpha^2} I - B_\rho R^{-1} B_\rho' \\ -\tilde{Q} & -\tilde{A}_\rho' \end{bmatrix} \end{aligned}$$

Hence as $\alpha \to \infty, H_\infty \to H_2$, $K_\infty \to K_2$ and the unified Hamiltonian converge towards the CT Hamiltonian as $\Delta \to 0$ [5].

5 Main Results

The previous scaled system realization is now used to formulate the H_2 and H_∞ control for the unified singularly perturbed systems. Consider the standard unified singularly perturbed system model [1],

Original Model:

$$\begin{aligned} \rho x_1(\tau) &= A_{11\rho} x_1(\tau) + A_{12\rho} x_2(\tau) + B_1 u(\tau) && (19a) \\ \epsilon x_2(\tau) &= A_{21\rho} x_1(\tau) + A_{22\rho} x_2(\tau) + B_{2\rho} u(\tau) && (19b) \\ y(\tau) &= C_{1\rho} x_1(\tau) + C_2 x_2(\tau) && (19c) \end{aligned}$$

Slow Subsystem:

$$\begin{aligned} \rho x_s(\tau) &= A_{\rho s} x_s(\tau) + B_{\rho s} u_s(\tau) && (20a) \\ y_s(\tau) &= C_\rho x_s(\tau) && (20b) \end{aligned}$$

where

$$\begin{aligned} A_{\rho s} &= A_{11\rho} - A_{12\rho} A_{22\rho}^{-1} A_{21\rho} \\ B_{\rho s} &= B_{1\rho} - A_{12\rho} A_{22\rho}^{-1} B_{2\rho} \\ C_{\rho s} &= C_{1\rho} - C_{2\rho} A_{22\rho}^{-1} A_{21\rho} \end{aligned}$$

Fast Subsystem:

$$\begin{aligned} \epsilon \rho x_2(\tau) &= A_{22\rho} x_f(\tau) + B_2 u_f(\tau) && (21a) \\ y_f(\tau) &= C_{2\rho} x_f(\tau) && (21b) \end{aligned}$$

where $x_f = x_2 - x_s, u_f = u - u_s$.

5.1 Slow Regulator Subsystem

The linear quadratic cost function for control based on the slow output and slow input is defined as:

$$J = S_0^\infty \left[y'(\tau) y(\tau) + u'(\tau) R u(\tau)\right] d\tau \quad (22)$$

The output feedback $y(\tau)$ can be separated into slow and fast components as follows:

$$y(\tau) = C_{1\rho} x_1(\tau) + C_{2\rho} x_2(\tau) = y_s(\tau) + y_f(\tau) + 0(\epsilon) \quad (23)$$

Then

$$\begin{aligned} y_s(\tau) &= C_{\rho s} x_s(\tau) + D_{\rho s} u_s(\tau) && (24) \\ y_f(\tau) &= C_f x_f(\tau) && (25) \end{aligned}$$

where

$$\begin{aligned} C_{\rho s} &= C_{1\rho} - C_{2\rho} A_{22\rho}^{-1} A_{21\rho} \\ D_{\rho s} &= -C_{2\rho} A_{22\rho}^{-1} B_{2\rho} \\ C_f &= C_{2\rho} \end{aligned}$$

The cost function for the slow function is:

$$\begin{aligned} J_s &= \frac{1}{2} S_0^\infty \Big[x_s'(\tau) C_{\rho s}' C_{\rho s} x_s(\tau) + 2 u_s'(\tau) D_{\rho s}' C_{\rho s} x_s(\tau) \\ &\quad + u_s'(\tau) R_s u_s(\tau)\Big] d\tau \end{aligned} \quad (26)$$

where

$$R_{\rho s} = R + D_{\rho s}' D_{\rho s}$$

The feedback control shown to minimize the above H_2 equivalent cost function is

$$u_{2s}(\tau) = K_{2s} x_s(\tau) \quad (27)$$

where

$$\begin{aligned} K_{2s} &= -\Big[(R_{\rho s} + \Delta B_{\rho s}' X_{2s} B_{\rho s})^{-1} \\ &\quad (B_{\rho s}' X_{2s} (I + \Delta A_{\rho s}) + D_{\rho s}' C_{\rho s})\Big] \end{aligned} \quad (28)$$

with

$$\begin{aligned} X_{2s} &= Ric(H_{2s}) \\ H_{2s} &= \begin{bmatrix} I & \Delta B_{\rho s} R_{\rho s}^{-1} B_{\rho s}' \\ 0 & I + \Delta \tilde{A}_{\rho s}' \end{bmatrix}^{-1} \begin{bmatrix} \tilde{A}_{\rho s} & -B_{\rho s} R_{\rho s}^{-1} B_\rho' \\ -\tilde{Q}_s & -\tilde{A}_{\rho s}' \end{bmatrix} \end{aligned} \quad (29)$$

where $\tilde{A}_{\rho s} = A_{\rho s} - B_{\rho s} R_{\rho s} D_{\rho s} C_{\rho s}'$ and $\tilde{Q}_s = -C_{\rho s}(I - D_{\rho s} R_{\rho s}^{-1} D_{\rho s}) C_{\rho s}'$. The frequency-domain representation for the slow subsystem model is then

$$G_{\rho s}(\gamma) = \left[\begin{array}{c|cc} \tilde{A}_{\rho s} & I & B_{\rho s} \\ \hline \begin{bmatrix} \tilde{Q}_s^{1/2} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ R_{\rho s}^{1/2} \end{bmatrix} \\ \hline I & 0 & 0 \end{array}\right]$$

The H_∞ solution for the general cost function is obtained by substituting the transformed and scaled system matrices into the results from [4]. The state feedback control for the slow subsystem is:

$$u_{\infty s}(\tau) = K_{\infty s} x_s(\tau) \quad (30)$$

where

$$
\begin{aligned}
K_{\infty s} &= -\Big[(R_{\rho s} + \Delta B'_{\rho s} X_{\infty s} B_{\rho s})^{-1} \\
&\quad (B'_{\rho s} X_{\infty s}(I + \Delta A_{\rho s}) + D'_{\rho s} C_{\rho s})\Big]
\end{aligned}
\tag{31}
$$

with

$$
\begin{aligned}
X_{\infty s} &= Ric(H_{\infty s}) \\
H_{\infty} &= \begin{bmatrix} I & -\Delta(\frac{1}{\alpha^2}I - B_{\rho s}R_{\rho s}^{-1}B'_{\rho s}) \\ 0 & (I + \Delta\tilde{A}'_{\rho s}) \end{bmatrix}^{-1} \\
&\quad \begin{bmatrix} \tilde{A}_{\rho s} & \frac{1}{\alpha^2}I - B_{\rho s}R_{\rho s}^{-1}B'_{\rho s} \\ -\tilde{Q}_s & -\tilde{A}'_{\rho s} \end{bmatrix}
\end{aligned}
\tag{32}
$$

We can easily see that as $\alpha \to \infty$, (32) $\to$ (29), $X_{\infty s} \to X_{2s}, K_{\infty s} \to K_{2s}$. Further, as $\Delta \to 0$ the unified $H_{\infty s}$ approaches the CT $H_{\infty s}$ results [5].

5.2 Fast Regulator Subsystem

In this subsection the solutions of the H_2 and H_∞ control are discussed for the fast subsystem. For the H_2 problem the cost function for the fast subsystem is:

$$
J = S_0^{\infty}\left[y'_f(\tau)y_f(\tau) + u'_f(\tau)Ru_f(\tau)\right]d\tau \tag{33}
$$

The feedback control shown to minimize the above is

$$
u_{2f}(\tau) = K_{2f}x_f(\tau), \tag{34}
$$

where

$$
\begin{aligned}
K_{2f} &= -(R + \Delta B'_{2\rho}X_{2f}B_{2\rho})^{-1} \\
&\quad B'_{2\rho}X_{2f}(I + \Delta A_{22\rho})x_f(\tau)
\end{aligned}
\tag{35}
$$

with

$$
\begin{aligned}
X_{2f} &= Ric(H_{2f}) \geq 0 \\
H_{2f} &= \begin{bmatrix} I & \Delta B_{2\rho}R^{-1}B'_{2\rho} \\ 0 & I + \Delta A'_{22\rho} \end{bmatrix}^{-1} \\
&\quad \begin{bmatrix} \tilde{A}_{22\rho} & -B_{2\rho}R^{-1}B'_{2\rho} \\ -Q_f & -A'_{22\rho} \end{bmatrix}
\end{aligned}
\tag{36}
$$

where $Q_f = Q$. The cost function for the fast subsystem for the H_∞ norm is then defined as:

$$
J_{\infty f} = \| T_{zw}(\gamma) \|_\infty = \| F_l(G_f(\gamma), K_{\infty f}) \|_\infty
$$

where the state-space realization for the fast subsystem, $G_f(\gamma)$ is

$$
G_{\rho f}(\gamma) = \left[\begin{array}{c|cc} A_{22\rho} & I_{A_{22\rho}} & B_{2\rho} \\ \hline \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \\ I_{A_{22\rho}} & 0 & 0 \end{array}\right]
$$

$$
\text{states} = \begin{bmatrix} x_f \\ w \\ u_f \end{bmatrix}; \ \text{outputs} = \begin{bmatrix} \rho x_f \\ x_2 \\ y_f \end{bmatrix} \tag{37}
$$

The state feedback for the fast subsystem H_∞ is

$$
u_{\infty f}(\tau) = K_{\infty f}x_f(\tau)
$$

where

$$
\begin{aligned}
K_{\infty f} &= -(R + \Delta B'_{2\rho}X_{\infty f}B_{2\rho})^{-1}B'_{2\rho}X_{\infty f}(I + \Delta A_{22\rho}) \\
K_{\infty f} &= Ric(H_{\infty f}) \geq 0 \\
H_{\infty f} &= \begin{bmatrix} I & -\Delta(\frac{1}{\alpha^2}I - B_{2\rho}R^{-1}B'_{2\rho}) \\ 0 & I + \Delta A_{22\rho}' \end{bmatrix}^{-1} \\
&\quad \begin{bmatrix} A_{22\rho} & \frac{1}{\alpha^2}I - B_{2\rho}R^{-1}B'_{2\rho} \\ -C'_2C_2 & -A'_{22\rho} \end{bmatrix}
\end{aligned}
\tag{38}
$$

Similarly for the fast subsystem, as $\alpha \to \infty$, (38) $\to$ (36), $X_{\infty f} \to X_{2f}, K_{\infty f} \to K_{2f}$. Further as $\Delta \to 0$ the unified $H_{\infty f}$ approaches the CT $H_{\infty f}$ results.

5.3 Composite Control

The composite control for the H_2 and H_∞ problem is given as:

$$
u_{2c}(\tau) = u_{2s}(\tau) + u_{2f}(\tau) = K_{2s}x_s(\tau) + K_{2f}x_f(\tau) \tag{39}
$$

$$
u_{\infty c}(\tau) = u_{\infty s} + u_{\infty f}(\tau) = K_{\infty s}x_s(\tau) + K_{\infty f}x_f(\tau) \tag{40}
$$

6 Aircraft Control Application

The formulation for H_∞ state-feedback control of the unified singularly perturbed system is now applied to control the longitudinal flight dynamics of an F-8 aircraft model. The longitudinal dynamics of the aircraft exhibits two-time scale properties identifiable by the phugoid (slow) mode and the short-period (fast) mode. H_∞ controllers are designed for the longitudinal axis dynamics of the aircraft for the cases of the full-order control, reduced control, and composite control.

6.1 Longitudinal F-8 aircraft model

The linearized small-perturbation longitudinal equations of motion and aerodynamic stability derivations are provided in [1]. The longitudinal F-8 aircraft model is for a flight condition of Mach 0.6 ($V_o = 620 ft\, s^{-1}$) altitude of 20 000 feet, and angle of attack of 0.078 rad. The state variables are v: velocity in ft s^{-1}); α: angle of attack (rad); q: pitch rate (rad s^{-1}); θ: pitch angle (rad); and the input variable is σ: stabilator deflection (rad). Here, the slow states are the forward air speed and pitch angle, while the fast states are the angle of attack and pitch rate.

With $\Delta = 0.25$, we obtain the space model from [1] as:

$$
A_{11\delta} = \begin{bmatrix} -0.3053 & -0.6536 \\ 1.4284 & -0.1223 \end{bmatrix}
$$

$$
A_{12\delta} = \begin{bmatrix} -0.8646 & 0.054 \\ -0.1635 & 0.0134 \end{bmatrix}
$$

$$
A_{21\delta} = \begin{bmatrix} -0.0464 & 0.004 \\ 0.0252 & -0.0018 \end{bmatrix}
$$

$$
A_{22\delta} = \begin{bmatrix} -0.4519 & 0.4029 \\ -1.9387 & -0.3159 \end{bmatrix}
$$

$$
B_{1\delta} = \begin{bmatrix} 1.3512 \\ -16.7794 \end{bmatrix} \quad B_{2\delta} = \begin{bmatrix} -0.2490 \\ -3.6615 \end{bmatrix}
$$

$$
C_{1\delta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad C_{2\delta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

6.2 Results

We provide the full-order design to provide a baseline for comparison with the reduced control and composite control designs. The H_∞ responses for the full-order system, composite control and the appropriate subsystem is shown in Figure 1,2,3 and 4. From Figure 1 and 2 (velocity and pitch angle), we notice that the composite controller does much better than the reduced order controllers. It is noticed that the slow controller, is not generally robust as the composite controller. However for the fast subsystem (Figure 3 and 4) the composite and the reduced order controller performances are satisfactory. The reduced and composite closed-loop responses are comparable with the full-order system.

7 Conclusion

The state space formulation for the H_∞ state feedback control for the unified singularly perturbed system was developed which includes scaling criteria for an equivalent ULQR and H_2 costs and transformations to accommodate the non-orthogonal terms. The formulation involves solving two reduced order problems and it was used to compute the reduced and composite H_∞ controller gains for the longitudinal dynamics of an F-8 aircraft. The reduced and composite closed-loop responses were then shown to be comparable with the full-order system. Here, a single unified method has been developed for a singular perturbation system replacing the previous two separate methods for the CT and DT systems. The unified H_∞ solution approaches the H_2 solution as α approaches infinity.

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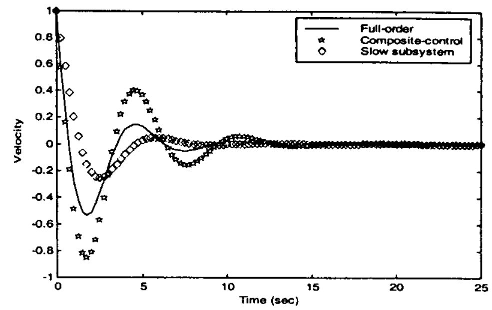

Figure 1: Closed-loop response of velocity of the F-8 aircraft

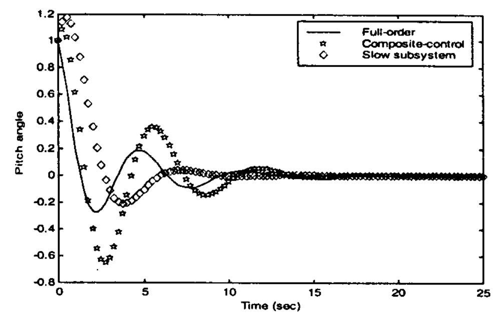

Figure 2: Closed-loop response of pitch angle of the F-8 aircraft

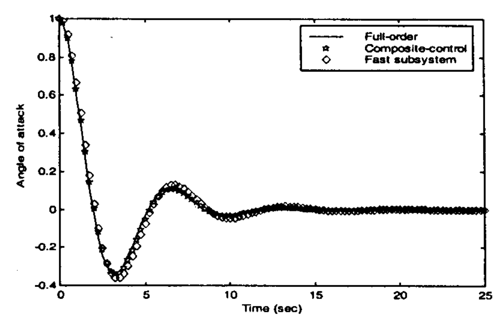

Figure 3: Closed-loop response of angle of attack of the F-8 aircraft

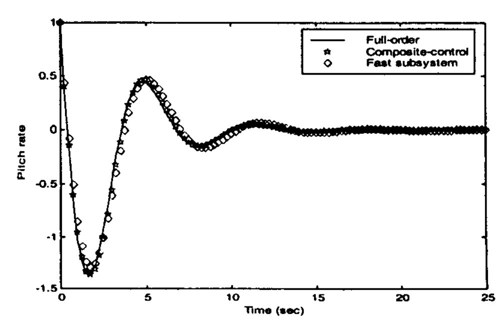

Figure 4: Closed-loop response of pitch rate of the F-8 aircraft