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LINEAR QUADRATIC REGULATOR WITH VARYING
FINITE TIME DURATIONS

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ABSTRACT

This paper considers the time-invariant linear quadratic regulator problem with varying finite time durations. It is shown that driving the state of a system along the same optimal trajectory but over a different time horizon requires a single Riccati gain matrix that is shifted appropriately in time. This result has significant ramifications for real-time implementation of optimal control systems.

INTRODUCTION

In the linear quadratic regulator problem one seeks the control law that drives a dynamic system from some initial state to a constant terminal state. This problem is typically cast using Pontryagin's minimum principle as a two-point boundary-value problem, where the state initial condition and co-state terminal condition are known. The solution of this problem in terms of the matrix differential Riccati equation is reported widely in the literature, and is developed in most modern control textbooks that present optimal control theory, e.g. (Sage and White, 1977; Owens, 1981, Nagrath and Gopal, 1983; Friedland, 1986).

Most often, the textbook presentations demonstrate the effect of increasing the terminal time and show the limiting case leading to the infinite-time regulator problem. For this case, the Riccati equation is reduced to the algebraic or steady-state matrix Riccati equation. The textbook presentations typically neglect discussion of the regulator problem when the terminal time, or time duration, assumes varying finite values. By default, it is presumed that in such cases the Riccati equation must be resolved for each problem with a unique terminal time or time interval.

This paper considers the time-invariant linear quadratic regulator problem with varying finite time durations. It is shown that varying the regulator's time duration does not require repeated calculations of the matrix Riccati equation. The main result of this paper can be summarized as follows: Driving the state of a system along the same optimal trajectory but over a different time horizon (or, equivalently, at a different speed) requires only a single feedback gain matrix that is shifted appropriately in time. In other words, once the Riccati equation is solved for a given time duration, it is not necessary to resolve it for a new problem with a different (shorter) time duration; the Riccati gain matrix can be obtained directly via a simple time-shift of the original Riccati matrix. This succinct result is not highlighted in the literature, and has significant

ramifications for real-time implementation of optimal control systems driven at various speeds, as borne out by a real-world example presented below.

BACKGROUND

In a linear quadratic regulator problem, the system is represented by a linear differential equation and the performance index is a quadratic functional of the state and control variables. Consider a system with dynamics that can be captured by the linear, time-invariant state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \quad (1)$$

where the state $\mathbf{x}(t) \in \mathbb{R}^n$, the control $\mathbf{u}(t) \in \mathbb{R}^m$, and the system matrix \mathbf{A} and the control influence matrix \mathbf{B} are dimensioned appropriately. Let the performance index be

$$J = \frac{1}{2} \mathbf{x}^T(t_f) \mathbf{S} \mathbf{x}(t_f) + \int_{t_0}^{t_f} \left[\frac{1}{2} \mathbf{x}^T(t) \mathbf{Q} \mathbf{x}(t) + \frac{1}{2} \mathbf{u}^T(t) \mathbf{R} \mathbf{u}(t) + \mathbf{x}^T(t) \mathbf{N} \mathbf{u}(t) \right] dt \quad (2)$$

One seeks the control $\mathbf{u}(t)$ that drives the system from an initial state ($\mathbf{x}(t_0)$) to a final state ($\mathbf{x}(t_f)$) while minimizing the performance index J . The solution of this linear quadratic regulator problem is known to have a closed-loop control form, where $\mathbf{u}(t)$ is expressed as a time weighted linear combination of $\mathbf{x}(t)$

$$\mathbf{u}(t) = \mathbf{K}_R(t) \mathbf{x}(t) \quad (3)$$

The elements of matrix $\mathbf{K}_R(t)$ are feedback coefficients or gains of the state variables. The matrix, referred to as the Riccati gain matrix (in some books, the Kalman gain matrix), can be expressed as:

$$\mathbf{K}_R(t) = -\mathbf{R}^{-1}[\mathbf{N}^T + \mathbf{B}^T \mathbf{P}(t)] \quad (4)$$

In equation (4) $\mathbf{P}(t)$ is a $(n \times n)$ symmetric matrix that satisfies the matrix differential Riccati equation:

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t) \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}(t) - \mathbf{P}(t) [\mathbf{A} - \mathbf{B} \mathbf{R}^{-1} \mathbf{N}^T] + [\mathbf{N} \mathbf{R}^{-1} \mathbf{B}^T - \mathbf{A}^T] \mathbf{P}(t) + \mathbf{N} \mathbf{R}^{-1} \mathbf{N}^T + \mathbf{Q} \quad (5)$$

with terminal condition:

$$P(t_f) = S \quad (6)$$

Since the Riccati gain matrix, $K_R(t)$, is independent of the state boundary conditions, it can be calculated off-line. This is a very attractive feature for real-time control applications.

Another important property of $K_R(t)$, with significant implications for real-time control, is the fact that it captures the complete character of the optimal trajectory solution for the time interval $(t_f - t_0)$ and any shorter interval.

LINEAR REGULATOR WITH VARYING TIME DURATION

Consider a Riccati gain matrix, $K_R(t)$, that drives the state of the system (A, B) from $\underline{x}(t_0)$ to $\underline{x}(t_f)$ in time interval $(t_f - t_0)$. Let $K_R'(t')$ be the Riccati gain matrix that drives the state of the same system (A, B) along the same optimal trajectory in a different time interval $(t_f' - t_0)$ (where $t_f' < t_f$).

Lemma: The Riccati gain matrix, $K_R'(t')$, is a subset of the gain matrix $K_R(t)$, or $K_R'(t') \in K_R(t)$.

Proof: The Riccati gain matrices, $K_R(t)$ and $K_R'(t')$, are derived from equations (4), (5), and (6) and are determined by the quantities (A, B, Q, R, N, S). At the terminal times, t_f and t_f' , the two gain functions have identical values:

$$K_R'(t_f') = K_R(t_f) = -R^{-1}[N^T + B^T S] \quad (7)$$

Since the matrix Riccati equation (5) is solved by integration backwards from the terminal time, the gain matrix for given (A, B, Q, R, N, S) at any time t is a function of the time, τ , remaining to complete the desired state trajectory, where $\tau = t_f - t$. Therefore,

$$K_R'(t') = K_R(t) \quad (8)$$

when the same remaining time intervals exist for the completion of the desired state trajectory, or when

$t' = t_f' - \tau$, and $t = t_f - \tau$. Hence, since $t_f' < t_f$ the Riccati gain matrix $K_R'(t')$ is a subset of $K_R(t)$.

Proposition: Let $K_R(t)$ be the Riccati gain matrix that optimally drives the state of the system (A, B) from $\underline{x}(t_0)$ to $\underline{x}(t_f)$ in time interval $(t_f - t_0)$. The Riccati gain matrix K_R' that drives the state along the same optimal trajectory in time interval $(t_f' - t_0)$ (where $t_f' < t_f$) can be expressed as:

$$K_R'(t) = K_R(t + (t_f - t_f')) \quad (9)$$

where $t_0 \leq t \leq t_f'$

This result expresses a very simple, yet important, property of the Riccati gain matrix. It enables the state of the system (A, B) to be driven along its optimal trajectory at various speeds while utilizing a single Riccati gain matrix with an appropriate time shift manipulation (equation (9)).

EXAMPLES

EXAMPLE 1:

This example is adapted from (Sage and White, 1977, Example 5.1-1). The system is described by:

$$\dot{x}(t) = -\frac{1}{2}x(t) + u(t)$$

The performance index is:

$$J = \int_0^{t_f} [x^2 + \frac{1}{2}u^2] dt$$

The Riccati gain $K_R(t)$ for this scalar system is found to be:

$$K_R(t) = \frac{1}{2} + \frac{3}{2} \tanh(1.5(t_f - t) - 0.346)$$

The time histories of $K_R(t)$ for $t_f = 1$ sec and $t_f = 10$ sec are shown in Figure 1. Let $K_R'(t)$ be the Riccati gain for $t_f = 1$ sec, and $K_R(t)$ be the gain for $t_f = 10$ sec. Then $K_R'(t)$ is a subset of $K_R(t)$ and can be expressed as:

$$K_R'(t) = K_R(t - 9) \quad \text{for } 0 \leq t \leq 1$$

This simple time shift manipulation can be used for evaluating the Riccati gain that drives the system state through its optimal trajectory at higher speeds. This result is not reported explicitly (to the authors' knowledge) in the open literature. It is, however, validated by inspection of figures, such as Figure 1, showing the Riccati gain for different terminal times. The importance of this seemingly standard observation is demonstrated in Example 2, where it is embedded in a practical control system design.

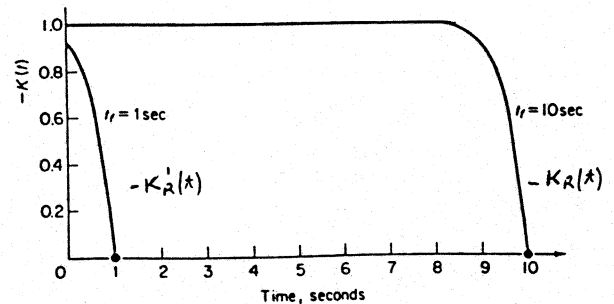


Figure 1. Riccati gain functions for terminal times of 1 and 10 sec. Adopted from (Sage, White, 1977, example 5.1-1).

EXAMPLE 2:

The second example considers a practical electromechanical system, the Penn State electric ventricular assist device (EVAD), shown in Figure 2. A model of this EVAD together with a mock loop circulatory system, shown in Figure 3, is derived in (Tasch, et. al., 1989).

The EVAD may operate in a loaded or unloaded mode. In the loaded mode, corresponding to the systolic condition, the pusher plate (see Figure 2) moves against a blood sac to eject blood and generate cardiac output. In the unloaded mode, or diastole, the pusher plate

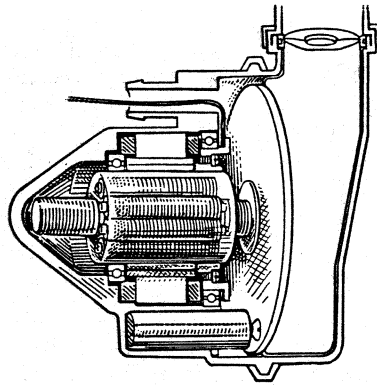


Figure 2. A Cross Section of The Cam Type Penn State Electric Ventricular Assist Device (EVAD).

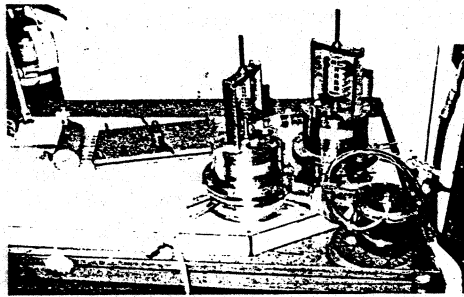


Figure 3. A Photograph of The EVAD together with a Mock Loop Circulatory System.

retracts to allow the sac to fill. The system has been modelled as having a single input and four state variables. The state equations in systole are:

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & -19.37 & -9.39 & 0 \\ 0 & 3.20 & -0.75 & 0.75 \\ 0 & 0 & 0.21 & -0.21 \end{Bmatrix} \mathbf{x}(t) + \begin{Bmatrix} 0 \\ 217 \\ 0 \\ 0 \end{Bmatrix} u(t)$$

The performance index is:

$$J = \frac{1}{2} \mathbf{x}^T(t_f) \begin{Bmatrix} 10 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix} \mathbf{x}(t_f) + \int_0^{t_f} [0.204 u(t)^2 + \mathbf{x}^T(t) \begin{Bmatrix} 0 \\ -0.0163 \\ 0 \\ 0 \end{Bmatrix} u(t)] dt$$

where the value of t_f is varied to adjust the EVAD's beat rate. For 50 beats per minute (bpm) and 50% systolic duration t_f is set to 0.6 sec. For 100 bpm (50% systolic duration) t_f is 0.3 sec. The time histories of the Riccati gains for 50 bpm, $K_R(t)$, and for 100 bpm, $K_R(t)$, are shown in Figure 4. These can be determined by integrating backwards the Riccati equation for each desired beat rate, or by using the main result of this paper. In this example

$$K_R'(t) = K_R(t - 0.3) \quad \text{for } 0 \leq t \leq 0.3 \text{ sec}$$

Furthermore, the Riccati gains of faster beat rates are subsets of the 50 bpm gains, and therefore the simple time-shift manipulation, expressed in equation (9), generates the Riccati gains that drive the EVAD in any desired beat rate.

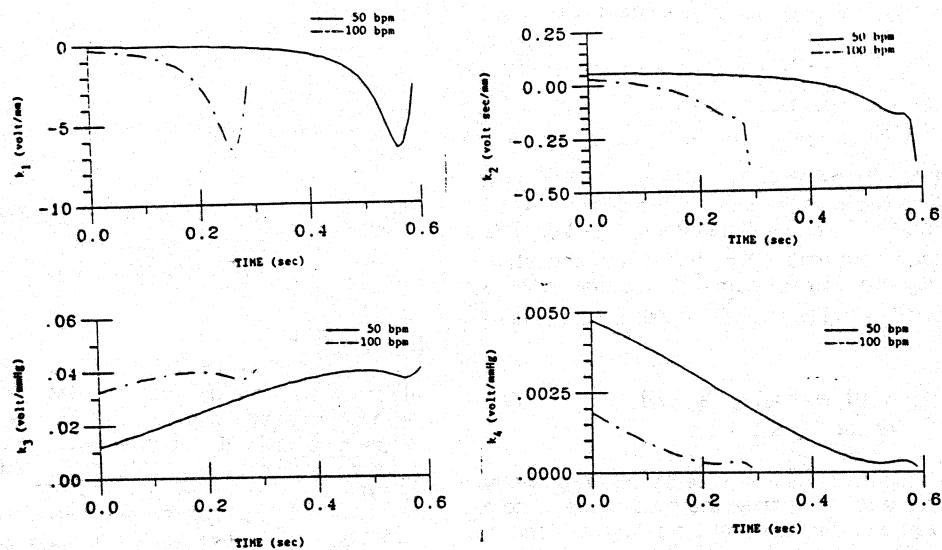


Figure 4. The Riccati Gain Matrices that Drive The EVAD at 50 and 100 (bpm).

DISCUSSION

Once optimal state and control trajectories have been computed for a linear quadratic regulator problem, it may be desirable to generate other optimal trajectories for the same problem but with different time intervals. Variations in time intervals, corresponding to changes in "speed", alter the state and control histories. While the original computations, namely the backwards integration of the Riccati equation, could be repeated for problems with different time intervals, there is a simpler way to determine the new optimal trajectories. Time-shift manipulations of the original Riccati gain matrix give the appropriate gain matrices directly.

This simple result is applicable only to time-invariant problems. For the time-varying case, equation (7) does not hold. For problems with time-varying systems and/or time-varying performance index, the Riccati gain matrix must be determined by solving the Riccati equation.

The main result of the linear regulator design, as presented above, assumes continuous time system, although a similar development readily follows for discrete time systems. (The discrete time exposition is not presented here.) The implementation of optimal control systems is based on digital computer control system logic. For example, the controller of Example 2 was built on such a sampled-data discrete time system. A sampled-data control law using a discrete time Riccati gain matrix, analogous to equation (9), was implemented and used to determine the optimal discrete-time trajectory. (Note: In such a system the control is held constant during each sampling interval. The resulting trajectory is close to the continuous-time optimum if the nominal control approximates its average value in the interval.)

Since the solution of the Riccati equation is guaranteed to be stable with infinite gain margin and 60° phase margin (Skelton, 1988), the time-shift technique advocated here is also stable for these conditions.

CONCLUSIONS

For the linear quadratic regulator problem, the optimal feedback law is linear and involves the solution of the Riccati equation. This paper shows that for time-invariant problems with finite but varying time intervals multiple solutions of the Riccati equation are not necessary. Rather, a more expeditious approach is to solve the Riccati equation once for a given time interval and then use time-shift manipulations of this Riccati gain matrix for all subsequent problems with different (shorter) time intervals.

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