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Technical Briefs

Linear Quadratic Regulator With Varying Finite Time Durations

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The optimal state trajectories of time-invariant linear quadratic regulator problems with different time horizons can be found from a single Riccati gain matrix shifted appropriately in time. This result has significant ramifications for real-time implementation of optimal controllers driving systems at various speeds.

Introdution

In the linear quadratic regulator problem, one seeks the control law that optimally drives a dynamic system from some initial state to a constant terminal state. The solution of this problem, credited to Kalman (1960, 1964), is cast in the terms of the matrix differential Riccati equation and is reported widely in the literature (e.g., Kirk, 1970; Kwakernaak and Sivan, 1972; Bryson and Ho, 1975; Sage and White, 1977; Owens, 1981; Friedland, 1986; Lewis, 1986). Most presentations demonstrate the effect of increasing the terminal time, and show the limiting case of the infinite-time regulator problem for which the Riccati equation is reduced to its algebraic form. Typically not addressed are multiple finite-time regulator problems where the terminal time, or time duration, is adjusted.

This technical note considers the time-invariant linear quadratic regulator problem with varying finite time durations. It is shown that the solutions of these finite time regulator problems do not require repeated calculations of the matrix differential Riccati equation. The main result of this paper can be summarized as follows: Driving the state of a system along the same optimal trajectory but over a shorter time horizon (or, equivalently, at a faster speed) requires computing a single feedback gain matrix that can be shifted appropriately in time. This result does not appear to be highlighted in the literature, and is important for real-time implementation of optimal control systems driven at various speeds, as borne out by a real-world example presented blow.

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Background

Consider a linear quadratic regulator problem with the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

where $\mathbf{x}(t)\epsilon\mathcal{R}^n$, $\mathbf{u}(t)\epsilon\mathcal{R}^m$, and $\mathbf{A}$ and $\mathbf{B}$ are dimensioned appropriately. The performance index is

$$J = \frac{1}{2}\mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \int_{t_0}^{t_f}\left[\frac{1}{2}\mathbf{x}T(t)\mathbf{Q}\mathbf{x}(t) + \frac{1}{2}\mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t) + \mathbf{x}^T(t)\mathbf{N}\mathbf{u}(t)\right]dt \tag{2}$$

where definiteness conditions exist on the weighting matrices $\mathbf{S}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{N}$. The problem is to determine the control $\mathbf{u}(t)$ that drives the system in finite time, $t_0 \le t \le t_f$, from an initial state, $\mathbf{x}(t_0)$, to a final state, $\mathbf{x}(t_f)$, that minimizes the performance index J. The solution assumes a closed-loop control form, where $\mathbf{u}(t)$ is expressed as

$$\mathbf{u}(t) = \mathbf{K}_R(t)\mathbf{x}(t) \tag{3}$$

$\mathbf{K}_R(t)$ is the Riccati gain matrix

$$\mathbf{K}_R(t) = -\mathbf{R}^{-1}[\mathbf{N}^T + \mathbf{B}^T\mathbf{P}(t)] \tag{4}$$

where $\mathbf{P}(t)$ is a (nxn) symmetric matrix that satisfies the matrix Riccati equation

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) - \mathbf{P}(t)[\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{N}^T] + [\mathbf{N}\mathbf{R}^{-1}\mathbf{B}^T - \mathbf{A}^T]\mathbf{P}(t) + \mathbf{N}\mathbf{R}^{-1}\mathbf{N}^T + \mathbf{Q} \tag{5}$$

with terminal condition

$$\mathbf{P}(t_f) = \mathbf{S} \tag{6}$$

Since $\mathbf{K}_R(t)$ is independent of the state boundary conditions, it can be calculated off-line and stored for table look-up. This feature is exploited often for real-time control applications. Another important property of $\mathbf{K}_R(t)$ is that it captures the complete character of the optimal trajectory solution for the interval $t_f - t_0$ and any shorter time interval.

Linear Regulator With Varying Finite Time

Consider a Riccati gain matrix, $\mathbf{K}_R(t)$, that drives the state of the system $(\mathbf{A}, \mathbf{B})$ from $\mathbf{x}(t_0)$ to $\mathbf{x}(t_f)$ in time interval $t_f - t_0$. Let $\mathbf{K}_R'(t')$ be the Riccati gain matrix that transfers the state of the system $(\mathbf{A}, \mathbf{B})$ along the same optimal trajectory in a shorter time interval $t_f' - t_0$, where $t_f' < t_f$. It can be shown that the Riccati gain matrix, $\mathbf{K}_R'(t')$, lies on the trajectory of the gain matrix $\mathbf{K}_R(t)$, i.e., $\mathbf{K}_R'(t')\epsilon\mathbf{K}_R(t)$.

To show that this is true note from Eqs. (4)–(6) that the Riccati gain matrices, $\mathbf{K}_R(t)$ and $\mathbf{K}_R'(t')$, are determined from the system $(\mathbf{A}, \mathbf{B})$ and performance index weighting $(\mathbf{S}, \mathbf{Q}, \mathbf{R}, \mathbf{N})$. At the terminal times t_f and t_f', the two gain functions have identical values

$$\mathbf{K}_R'(t_f') = \mathbf{K}_R(t_f) = -\mathbf{R}^{-1}[{}^T + \mathbf{B}^T\mathbf{S}] \qquad (7)$$

Since the matrix Riccate Eq. (5) is solved by integration backwards from the terminal time, the gain matrix for given (**A**, **B**, **S**, **Q**, **R**, **N**) at any time t is a function of the time, τ, remaining to complete the desired state trajectory, where $\tau = t_f - t$. Therefore, when $t' = t_f' - \tau$ and $t = t_f - \tau$ the gain matrices are equal.

$$\mathbf{K}_R'(t') = \mathbf{K}_R(t) \qquad (8)$$

Thus, since $t_f' < t_f$ the matrix $\mathbf{K}_R'(t')$ is a subset of $\mathbf{K}_R(t)$.

Furthermore, if $\mathbf{K}_R(t)$ is the Riccati gain matrix that optimally drives the state of the system (**A**, **B**) from $\mathbf{x}(t_0)$ to $\mathbf{x}(t_f)$ in time interval $t_f - t_0$, then the Riccati gain matrix that transfers the state along the same optimal trajectory in time interval $t_f' - t_0$, where $t_f' < t_f$, can be expressed as

$$\mathbf{K}_R'(t) = \mathbf{K}_R(t - (t_f - t_f')) \text{ where } t_0 \le t \le t_f' \qquad (9)$$

This result summarizes a very simple, yet significant, property of the Riccati gain matrix. It enables the state of the system (**A**, **B**) to be driven along an optimal trajectory at various speeds by utilizing a single Riccati gain matrix ($\mathbf{K}_R(t)$) delayed by appropriate time shifts.

Example

Example 1. In this example, adapted from (Sage and White, 1977, Example 5.1-1), the system model is

$$\dot{x}(t) = -\frac{1}{2}x(t) + u(t)$$

and the performance index is

$$J = \int_0^{t_f} [x^2 + \frac{1}{2}u^2]dt$$

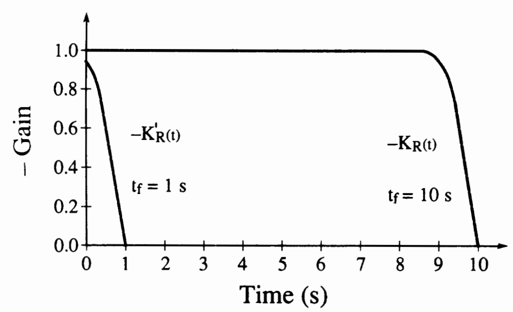

Fig. 1 Riccati gain functions for terminal times of 1 s and 10 s. (Adopted from Sage and White, 1977, example 5.1-1.)

For this problem, the solution of the Riccati equation (which reduces to a scalar differential equation) gives the gain

$$K_R(t) = 0.5 + 1.5 \tanh(1.5(t_f - t) - 0.346)$$

The time histories of $K_R(t)$ for $t_f = 1$ s and $t_f = 10$ s are shown in Fig. 1. Denoting $K_R'(t)$ as the Riccati gain for $t_f = 1$ s and $K_R(t)$ as the gain for $t_f = 10$ s, then $K_R'(t)$ can be found from $K_R(t)$ according to $K_R'(t) = K_R(t-9)$ for $0 \le t \le 1$ s. In summary, a time shift manipulation of K_R(t) gives the Riccati gain that drives the system through its optimal state trajectory at any higher speed.

Example 2. The time shift property was exploited in controlling the Penn State electric ventricular assist device (EVAD), the model of which is derived in (Tasch et al., 1989).

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -19.37 & -9.39 & 0 \\ 0 & 3.20 & -0.75 & 0.75 \\ 0 & 0 & 0.21 & -0.21 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ 217 \\ 0 \\ 0 \end{pmatrix} u(t)$$

$$J = \frac{1}{2}\mathbf{x}^T(t_f) \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x}(t_f) + \int_0^{t_f} [0.204u(t)^2$$

$$+ \mathbf{x}^T(t) \begin{pmatrix} 0 \\ -0.0163 \\ 0 \\ 0 \end{pmatrix} u(t)]dt$$

The performance index reflects the desire to operate the EVAD with minimal electrical energy consumption. The value of t_f is varied to adjust the beat rate. For 50 beats per minute (bpm) t_f is set to 0.6s, and for 100 bpm t_f is 0.3 s. The time histories of the Riccati gains $\mathbf{K}_R(t)$ and $\mathbf{K}_R'(t)$ for 50 bpm and 100 bpm, respectively, are shown in Fig. 2. For each desired beat rate the controller computed the gain matrix according to $\mathbf{K}_R'(t) = \mathbf{K}_R(t - 0.3)$ for $0 \le t \le 0.3$ s.

Closing

Once optimal state and control trajectories have been computed for a linear quadratic regulator problem[3], it may be

[3] The deterministic linear optimal regulator problem always has a unique solution (Kwakernaak and Sivan, 1971).

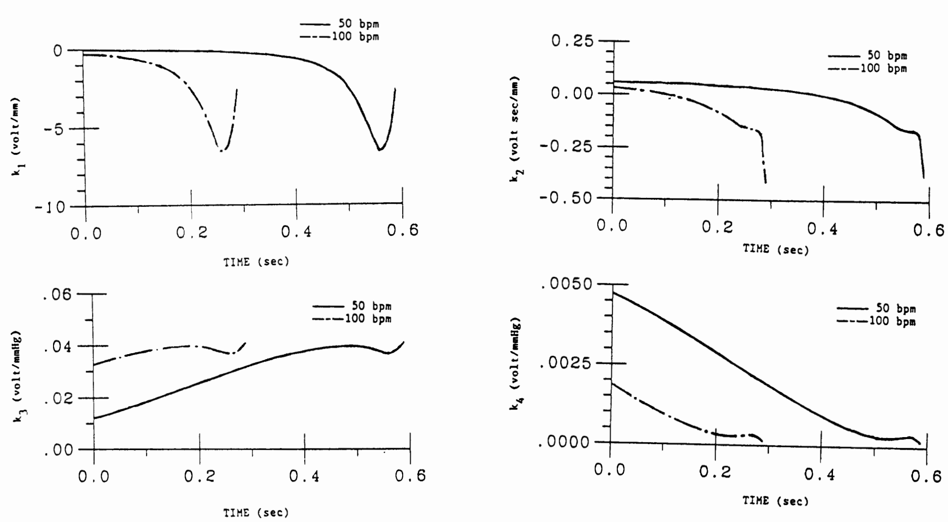

Fig. 2 Riccati gain matrices that drive EVAD at 50 and 100 bpm

desirable to generate other optimal trajectories for the same problem but with different time intervals. While the backwards integration of the Riccati equation could be repeated for different time intervals, a simple time-shift manipulation gives the appropriate gain matrices directly. This result is applicable to (*i*) time-invariant systems only, and (*ii*) continuous and discrete time systems (the formulation for discrete time systems readily follows).

In conclusion, the optimal feedback law of the linear quadratic regulator problem is linear and involves the solution of the Riccati equation. This note shows that for time-invariant problems with finite but varying time intervals multiple solutions of the Riccati equation are not necessary. Time shift manipulations of the Riccati gain matrix yield the gain matrices for regulator problems with shorter time intervals.

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A Frequency-Domain Suboptimal Controller Design Methodology for Linear Time-Invariant Systems With Controller Structural Constraints

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1 Introduction

For linear time-invariant dynamic systems with quadratic criteria, it is well known that the linear optimal controller is constituted by feedback of system states, in which the feedback gains are determined through the solution of a Riccati equation. However, this optimal controller is often difficult, if not impossible, to implement because the full system states are generally not available in reality, and are too complicated to estimate on-line. In many engineering applications, especially in view of heightening the system benefit-cost ratio, it is sometimes desirable to design a controller based on the available system measurements. This is referred to as constrained suboptimal control problem (Kosut, 1970) and is often encountered in practice.

The problem of output feedback controller designs for linear time invariant systems was discussed by Man (1970), Dabke (1970), Kosut (1970), Paul (1980), Ly (1985), and Zheng (1989). Two approximate methods have been developed by Kosut. In this paper, a general frequency-domain method is proposed for the design of suboptimal controllers under either deterministic or stochastic input conditions. The method can also be applied to the design of output feedback controllers.

2 Problem Formulation

We consider the following time-invariant dynamic system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{D}\mathbf{w}(t) \tag{1}$$

where $\mathbf{x} \in R^{n\times 1}$, $\mathbf{u} \in R^{m\times 1}$ are the system state and control vectors, respectively; $\mathbf{A} \in R^{n\times n}$, $\mathbf{B} \in R^{n\times m}$, $\mathbf{D} \in R^{n\times r}$ are constant matrices; and $\mathbf{w} \in R^{r\times 1}$ is a zero-mean white noise process with spectral density

$$E[\mathbf{w}(t)\,\mathbf{w}^T(\tau)] = \mathbf{W}_0\delta(t-\tau) \tag{2}$$

The system output vector $\mathbf{y} \in R^{l\times 1}$ is

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{3}$$

where $\mathbf{C} \in R^{l\times n}$ is a constant matrix.

The quadratic performance criterion is

$$J = \lim_{T\to\infty} E\left[\frac{1}{T}\int_0^T (\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,dt\right] \tag{4}$$

It is known that if $\mathbf{Q}$ is semi-positive definite, $\mathbf{R}$ is positive definite, and $(\mathbf{A}, \mathbf{B})$ is controllable, the optimal linear control $\mathbf{u}^0(t)$ is given by (Kwakernaak and Sivan, 1972)

$$\mathbf{u}^0(t) = -\mathbf{F}^0\mathbf{x}(t) \tag{5}$$

where

$$\mathbf{F}^0 = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \tag{6}$$

and $\mathbf{P}$ is the solution of the Riccati equation:

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{Q}\mathbf{C} = 0 \tag{7}$$

Although this optimal linear control law is desirable, it can rarely be used in a practical situation because the full state information $\mathbf{x}(t)$ is difficult to obtain in real systems. Instead, the measured quantity $\mathbf{z}(t)$ must be used directly as feedback:

$$\mathbf{u}(t) = -\mathbf{K}^T\mathbf{z}(t) \tag{8}$$

where $\mathbf{z} \in R^{s\times 1}$ is the measurement vector

$$\mathbf{z}(t) = \mathbf{M}\mathbf{x}(t) \tag{9}$$

and $\mathbf{K}^T \in R^{m\times s}$, $\mathbf{M} \in R^{s\times n}$ are constant matrices.

Combining (8) and (9) produces

$$\mathbf{u}(t) = -\mathbf{K}^T\mathbf{M}\mathbf{x}(t) = -\mathbf{F}\mathbf{x}(t) \tag{10}$$

where

$$\mathbf{F} = \mathbf{K}^T\mathbf{M} \quad \in R^{m\times n} \tag{11}$$

In a more general form, the multiple structural constrained control law can be specified as (Kosut, 1970)

$$\mathbf{u}_i(t) = -\mathbf{K}_i^T\mathbf{z}_i(t) = -\mathbf{F}_i\mathbf{x}(t) \tag{12}$$

where

$$\mathbf{z}_i(t) = \mathbf{M}_i\mathbf{x}(t) \in R^{s_i\times 1}, \quad (\mathbf{M}_i \in R^{s_i\times n}) \tag{13}$$

$$\mathbf{F}_i = \mathbf{K}_i^T\mathbf{M}_i \in R^{1\times n}, \quad (\mathbf{K}_i^T \in R^{1\times s_i}) \tag{14}$$

In this paper, the more general multiple constraints are taken into account. The problem now turns into the determination of $\mathbf{K}_i$ in the new feedback gain $\mathbf{F}_i$ for each controller in (12) such that J of (4) is a minimum.

Although many numerical methods such as gradient search

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