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LINEAR QUADRATIC OPTIMAL CONTROL BY CHEBYSHEV-BASED STATE REPRESENTATION

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Abstract

This paper presents a computationally efficient method for solving optimal control problems involving unconstrained linear time-invariant dynamic systems with quadratic performance indices. In the proposed method, the difference between each state variable and its initial condition is represented by a finite-term shifted Chebyshev series. The representation leads to a system of linear algebraic equations as the necessary condition of optimality. The results of simulation studies demonstrate that the Chebyshev-based method offers computational advantages relative to a standard Riccati-based and/or transition matrix methods.

Introduction

The optimal control of linear, lumped parameter models of dynamic systems is one of the principal "state space" design problems. In this problem the optimal control trajectories and associated state trajectories are sought which give the best tradeoff between performance and cost of control. In the general formulation using variational methods, the optimality condition of this problem is cast as a two-point boundary-value problem (TPBVP). One of the most well-known solution approaches is the Hamilton-Jacobi approach which converts the TPBVP to a terminal value problem involving a matrix differential Riccati equation. The Riccati equation gives the optimal solution in closed-loop form making it a preferred approach for physical implementation, although it is computationally intensive and sometimes difficult to employ in solving high order systems.

A preferred alternative for the optimal control solution of time-invariant problems is the open-loop transition matrix approach (Speyer, 1986). Typically, the transition matrix approach converts the TPBVP into an initial value problem. The transition matrix approach is also susceptible to numerical problems in determining the optimal control of high order systems (Yen and Nagurka, 1990). In particular, numerical instabilities, attributed principally to the error associated with the computation of large dimension state transition matrices, can occur. An accurate and computationally streamlined approach for calculation of state transition matrices of high order systems remains a research challenge (Moler and Loan, 1978).

In contrast to Riccati-based and transition matrix methods, approximate solution strategies, namely trajectory parameterization methods, have been investigated. In general, these approaches approximate the control, state, and/or co-state trajectories by finite-term orthogonal functions whose unknown coefficient values are sought giving a near optimal (or sub-optimal) solution. For example, approaches employing functions such as Walsh (Chen and Hsiao, 1975), block-pulse (Hsu and Cheng, 1981), Laguerre (Shih, Kung and Chao, 1986), Chebyshev (Paraskevopoulos, 1983; Vlassenbroeck and Van Dooren, 1988), and Fourier (Chung, 1987) have been suggested. Like the state transition matrix approach, many of these approaches employ algorithms that convert the TPBVP into an initial value problem. The initial value problem is then integrated with respect to time whose state and co-state vectors are then approximated by truncated orthogonal series. This technique reduces the initial value problem to a static optimization problem represented by algebraic equations. The truncation of the orthogonal series results in errors, which can be minimized by including more terms. However, the transition matrix (needed to convert the TPBVP to an initial value problem) must still be evaluated which, as mentioned above, can cause instability problems in high order systems.

A premise of on-going research is the utility of computational tools for solving optimal control problems via state parameterization. An advantage of state parameterization is that the state initial condition can be satisfied directly. A second advantage is that the state equation can be treated as a set of algebraic equations in determining the corresponding control trajectory. This assumes that there are no constraints on the control structure preventing an arbitrary representation of the state trajectory from being achieved.

This paper extends the work of (Yen and Nagurka, 1988) for solving optimal control problems via Fourier-based state parameterization. Their work has shown that a Fourier-based state approximation offers an accurate, computationally efficient, and robust methodology for solving linear quadratic (LQ) optimal control problems relative to standard methods. For systems with different numbers of state variables and control variables, artificial control variables were introduced to overcome the potential difficulty of trajectory inadmissibility. These physically non-existent variables are driven small by being heavily penalized in the performance index. The particular focus of this paper is to explore a parameterization approach based on a finite-term Chebyshev representation of the state trajectory. Chebyshev functions can nearly uniformly approximate a broad class of functions, making them computationally attractive (Vlassenbroeck and Van Dooren, 1988). It is shown that the necessary condition of optimality can be derived as a system of linear algebraic equations from which an unknown state parameter vector can be solved. The method is accurate and computationally very attractive, especially for high-order systems.

Chebyshev-based Approach

Problem Statement

The LQ optimal control problem involves finding the control $\mathbf{u}(t)$ and the corresponding state $\mathbf{x}(t)$ in the time interval [0,T] that minimizes the quadratic performance index L,

$$L = L_1 + L_2 \tag{1}$$

where

$$L_1 = \mathbf{x}^T(T)\mathbf{H}\mathbf{x}(T) + \mathbf{h}^T\mathbf{x}(T) \tag{2}$$

$$L_2 = \int_0^T [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t)+\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)+\mathbf{x}^T(t)\mathbf{S}(t)\mathbf{u}(t)+\mathbf{q}^T(t)\mathbf{x}(t)+\mathbf{r}^T(t)\mathbf{u}(t)]dt \tag{3}$$

for the linear dynamic system with the state-space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{4}$$

with known initial condition $\mathbf{x}(0)=\mathbf{x}_0$. The state vector $\mathbf{x}$ is N×1, the control vector $\mathbf{u}$ is M×1, the system matrix $\mathbf{A}$ is N×N, and the control influence matrix $\mathbf{B}$ is N×M. It is assumed that weighting matrices $\mathbf{H}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ and weighting vectors $\mathbf{h}$, $\mathbf{q}$ and $\mathbf{r}$ have appropriate dimensions, and that $\mathbf{H}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ are real and symmetric with $\mathbf{H}$ and $\mathbf{Q}$ being positive-semidefinite and $\mathbf{R}$ being positive definite.

Chebyshev Polynomials

In the parameterization promoted below, the basis functions of the approximated state vector include shifted Chebyshev polynomials. In general, Chebyshev polynomials are defined for the interval $\xi \in [-1,1]$ and have the following analytical form:

$$\varphi_k(\xi) = \cos(k\cos^{-1}\xi) = \sum_{i=0}^{[k/2]} (-1)^i \frac{k!}{(2i)!(k-2i)!}(1-\xi)^i \xi^{k-2i} \quad , \quad k = 0, 1, 2, \ldots \tag{5}$$

where the notation [k/2] means the greatest integer smaller than k/2. In shifted Chebyshev polynomials the domain of the Chebyshev polynomials is transformed to values between 0 and T by introducing the change of variables $\xi = 2\,t\,/\,T - 1$ giving

$$\psi_k(t) = \varphi_k(\xi) = \varphi_k(2\tau - 1) \tag{6}$$

where nondimensional time $\tau = t\,/\,T$. From Equation (6) the first few shifted Chebyshev polynomials are

$$\psi_0(t) = 1 \quad , \quad \psi_1(t) = 2\tau - 1 \quad , \quad \psi_2(t) = 8\tau^2 - 8\tau + 1$$
$$\psi_3(t) = 32\tau^3 - 48\tau^2 + 18\tau - 1 \quad , \quad \psi_4(t) = 128\tau^4 - 256\tau^3 + 160\tau^2 - 32\tau + 1 \tag{7a-e}$$

The initial and final values of the shifted Chebyshev polynomial and its first time derivative can be obtained as

$$\psi_k(0) = (-1)^k \quad , \quad \dot{\psi}_k(0) = (-1)^{k+1}(2k^2/\,T)$$
$$\psi_k(T) = 1 \quad , \quad \dot{\psi}_k(T) = 2k^2/\,T \tag{8a-d}$$

State Parameterization

The LQ optimal control problem can be converted into a quadratic programming (QP) problem by approximating each of the N state variables $x_n(t)$ by the summation of the initial condition and a K term series.

$$x_n(t) = x_{n0} + \sum_{k=1}^{K} c_k(t)\, y_{nk} \tag{9}$$

where $x_{n0}=x_n(0)$ and y_{nk} (k=1,2,...,K and n=1,2,...,N) is the k-th unknown coefficient of the basis function $c_k(t)$ for the n-th state variable. A variety of basis functions can be used as long as the state initial condition is satisfied. Here, a shifted Chebyshev-series modified by the addition of a term such that $c_k(0) = 0$ is proposed; *i.e.*, the basis function is

$$c_k(t) = \psi_k(t) + (-1)^{k-1} \quad , \quad k = 1, 2, \cdots, K \tag{10}$$

Equation (9) can be written alternatively as

$$x_n(t) = x_{n0} + \mathbf{c}^T(t)\,\mathbf{y}_n \tag{11}$$

where $\mathbf{c}^T(t)$ and $\mathbf{y}_n$ are

$$\mathbf{c}^T(t) = [c_1(t) \;\; c_2(t) \;\; \cdots \;\; c_K(t)] \tag{12}$$

$$\mathbf{y}_n = [y_{n1} \;\; y_{n2} \;\; \cdots \;\; y_{nK}]^T \tag{13}$$

In Equation (13) $\mathbf{y}_n$ is a state parameter vector (containing the unknown coefficients) for the n-th state variable.

The state vector containing the N state variables can be written in terms of a full state parameter vector $\mathbf{y}$, *i.e.*,

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{C}(t)\mathbf{y} \tag{14}$$

where

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{c}^T(t) & & & 0 \\ & \mathbf{c}^T(t) & & \\ & & \ddots & \\ 0 & & & \mathbf{c}^T(t) \end{bmatrix}_{N\times NK} \tag{15}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} [y_{11} & y_{12} & \cdots & y_{1K}]^T \\ [y_{21} & y_{22} & \cdots & y_{2K}]^T \\ & \vdots & & \\ [y_{N1} & y_{N2} & \cdots & y_{NK}]^T \end{bmatrix}_{NK\times 1} \tag{16}$$

Similarly, the state rate vector can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{D}(t)\mathbf{y} \tag{17}$$

where

$$\mathbf{D}(t) = \dot{\mathbf{C}}(t) = \begin{bmatrix} \mathbf{d}^T(t) & & & 0 \\ & \mathbf{d}^T(t) & & \\ & & \ddots & \\ 0 & & & \mathbf{d}^T(t) \end{bmatrix}_{N\times NK} \tag{18}$$

$$\mathbf{d}^T(t) = [\,\dot{c}_1(t)\;\dot{c}_2(t)\cdots\dot{c}_K(t)\,] = [\,\dot{\psi}_1(t)\;\dot{\psi}_2(t)\cdots\dot{\psi}_K(t)\,] \tag{19}$$

The control vector $\mathbf{u}(t)$ can also be expressed as a function of $\mathbf{y}$. From Equations (4), (14), and (17),

$$\mathbf{u}(t) = \left[\mathbf{B}^{-1}(t)\mathbf{D}(t) - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{C}(t)\right]\mathbf{y} - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{x}_0 \tag{20}$$

Equation (20) assumes that $\mathbf{B}^{-1}$ exists and implies that the lengths of the state and control vectors are the same (*i.e.*, M=N). By employing artificial control variables, this requirement is later relaxed. (See section on General Linear Systems.)

Approximation of Performance Index

By substituting the parameterized state vector and control vector into the performance index, the performance index can be approximated as a function of the state parameter vector **y**. First, Equation (14) with t=T is substituted into Equation (2) to give the cost L_1 as

$$L_1 = \mathbf{y}^T\left[\mathbf{H}\otimes\mathbf{c}(T)\mathbf{c}^T(T)\right]\mathbf{y} + \mathbf{y}^T\left[(2\mathbf{H}\mathbf{x}_0+\mathbf{h})\otimes\mathbf{c}^T(T)\right] + \mathbf{x}_0^T(\mathbf{H}\mathbf{x}_0+\mathbf{h}) \tag{21}$$

From Equations (14) and (20) the integrand of Equation (3) can be expressed as a function of the parameter vector **y**, *i.e.*,

$$\mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}+\mathbf{x}^T\mathbf{S}\mathbf{u}+\mathbf{q}^T\mathbf{x}+\mathbf{r}^T\mathbf{u} = \mathbf{y}^T\mathbf{P}\mathbf{y}+\mathbf{y}^T\mathbf{p}+\mathbf{x}_0^T\mathbf{p}_0 \tag{22}$$

where

$$\begin{aligned} \mathbf{P} &= \mathbf{F}_1\otimes\mathbf{c}\mathbf{c}^T+\mathbf{F}_2\otimes\mathbf{d}\mathbf{d}^T+\mathbf{F}_3\otimes\mathbf{d}\mathbf{c}^T \\ \mathbf{p} &= (2\mathbf{F}_1\mathbf{x}_0+\mathbf{f}_1)\otimes\mathbf{c}+(\mathbf{F}_3\mathbf{x}_0+\mathbf{f}_2)\otimes\mathbf{d} \\ \mathbf{p}_0 &= \mathbf{F}_1\mathbf{x}_0+\mathbf{f}_1 \end{aligned} \tag{23a-c}$$

where $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ are N×N matrices and $\mathbf{f}_1$ and $\mathbf{f}_2$ are N×1 vectors given by

$$\begin{aligned} &\mathbf{F}_1 = \mathbf{Q}+\mathbf{A}^T\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{A}-\mathbf{S}\mathbf{B}^{-1}\mathbf{A}, \qquad \mathbf{F}_2 = \mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1} \\ &\mathbf{F}_3 = -2\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{A}+\mathbf{B}^{-T}\mathbf{S}, \qquad \mathbf{f}_1 = \mathbf{q}-\mathbf{A}^T\mathbf{B}^{-T}\mathbf{r}, \qquad \mathbf{f}_2 = \mathbf{B}^{-T}\mathbf{r} \end{aligned} \tag{23a-e}$$

and superscript $-T$ denotes inverse transpose. In Equation (23a-c), **P** is an NK×NK matrix, **p** and $\mathbf{p}_0$ are an NK×1 vector, and ⊗ is a Kronecker product sign (Brewer, 1978), *e.g.*,

$$\mathbf{V}\otimes\mathbf{W} = \begin{bmatrix} V_{11}\mathbf{W} & \cdots & V_{1n}\mathbf{W} \\ V_{21}\mathbf{W} & & \vdots \\ \vdots & & \vdots \\ V_{n1}\mathbf{W} & \cdots & V_{nn}\mathbf{W} \end{bmatrix} \tag{25}$$

where **V** is an n×n matrix and **W** is an arbitrary matrix. Thus, from Equation (22), the integral part of the performance index can be expressed as

$$L_2 = \int_0^T \left(\mathbf{y}^T\mathbf{P}\mathbf{y} + \mathbf{y}^T\mathbf{p} + \mathbf{x}_0^T\mathbf{p}_0\right)dt = \mathbf{y}^T\mathbf{P}^*\mathbf{y} + \mathbf{y}^T\mathbf{p}^* + \mathbf{x}_0^T\mathbf{p}_0^* \tag{26}$$

where

$$\mathbf{P}^* = \int_0^T \mathbf{P}\,dt \qquad \mathbf{p}^* = \int_0^T \mathbf{p}\,dt \qquad \mathbf{p}_0^* = \int_0^T \mathbf{p}_0\,dt \tag{27a-c}$$

can be integrated numerically. Combining Equations (21) and (26) gives the performance index L as a function of the state parameter vector $\mathbf{y}$, *i.e.*,

$$L = \mathbf{y}^T\Omega\mathbf{y} + \mathbf{y}^T\omega + \mathbf{x}_0^T\left[\mathbf{H}\mathbf{x}_0 + \mathbf{h} + \mathbf{p}_0^*\right] \tag{28}$$

where

$$\Omega = \mathbf{H}\otimes\mathbf{c}(T)\mathbf{c}(T)^T + \mathbf{P}^* \quad , \qquad \omega = (2\mathbf{H}\mathbf{x}_0+\mathbf{h})\otimes\mathbf{c}^T(T) + \mathbf{p}^* \tag{29a-b}$$

For time-invariant problems, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are constants, and Equations (27a-c) can be rewritten as

$$\mathbf{P}^* = \mathbf{F}_1\otimes\left[\int_0^T (\mathbf{c}\,\mathbf{c}^T)dt\right] + \mathbf{F}_2\otimes\left[\int_0^T (\mathbf{d}\,\mathbf{d}^T)dt\right] + \mathbf{F}_3\otimes\left[\int_0^T (\mathbf{d}\,\mathbf{c}^T)dt\right]$$

$$\mathbf{p}^* = (2\mathbf{F}_1\mathbf{x}_0+\mathbf{f}_1)\otimes\left[\int_0^T \mathbf{c}\,dt\right] + (\mathbf{F}_3\mathbf{x}_0+\mathbf{f}_2)\otimes\left[\int_0^T \mathbf{d}\,dt\right] \tag{30a-c}$$

$$\mathbf{p}_0^* = T(\mathbf{F}_1\mathbf{x}_0+\mathbf{f}_1)$$

The terms in the brackets can be evaluated numerically or derived in closed-form. For example, the first integral term of Equation (30b) can be derived as

$$\int_0^T c_i\,dt = \int_0^T \psi_i(t)\,dt + (-1)^{k-1}T = \frac{T}{2}\alpha_{0i} + (-1)^{k-1}T \tag{31}$$

where

$$\alpha_{01} = 0 \quad \text{and} \quad \alpha_{0i} = \frac{1+(-1)^i}{1-i^2}, \quad i\neq 1 \tag{32}$$

Optimality Condition

The optimal control problem now can be viewed as the search for the unknown coefficients of the state parameter vector $\mathbf{y}$ that minimize the parameterized performance index of Equation (28). The necessary condition of optimality can be obtained by differentiating the performance index with respect to the unknown vector $\mathbf{y}$. This leads to

$$\left(\Omega + \Omega^T\right)\mathbf{y} = -\omega \tag{33}$$

which represents a system of linear algebraic equations from which the unknown vector $\mathbf{y}$ can be solved.

General Linear Systems

To apply the Chebyshev-based approach to general linear systems which have fewer control variables than state variables, the state-space model of Equation (4) is modified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}'(t)\mathbf{u}'(t) \tag{34}$$

where

$$\mathbf{B}'(t) = \mathbf{B}'_{N\times N} = \begin{bmatrix} \mathbf{I}_{(N\text{-}M)\times(N\text{-}M)} & \\ & \mathbf{B}_{N\times M} \\ \mathbf{0}_{M\times(N\text{-}M)} & \end{bmatrix} \tag{35}$$

$$\mathbf{u}'(t) = \mathbf{u}'_{N\times 1} = \begin{bmatrix} \hat{\mathbf{u}}_{(N\text{-}M)\times 1} \\ \\ \mathbf{u}_{M\times 1} \end{bmatrix} \tag{36}$$

where $\hat{\mathbf{u}}$ is an artificial (*i.e.*, fictitious) control vector.

It can be guaranteed that **B**' is invertible if the last M rows of **B** are nonsingular. However, if the last M rows are singular, the first (N-M) columns of **B**' in Equation (35) can be modified to make it invertible. In order to predict the optimal solution, the performance index is modified to

$$L' = L_1 + L'_2 \tag{37}$$

where

$$L'_2 = \int_0^T [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t)+\mathbf{u}'^T(t)\mathbf{R}'(t)\mathbf{u}'(t)+\mathbf{x}^T(t)\mathbf{S}'(t)\mathbf{u}'(t)+\mathbf{q}^T(t)\mathbf{x}(t)+\mathbf{r}'^T(t)\mathbf{u}'(t)]dt \tag{38}$$

with

$$\mathbf{R}'(t) = \mathbf{R}'_{N\times N} = \begin{bmatrix} \rho\mathbf{I}_{(N\text{-}M)\times(N\text{-}M)} & \mathbf{0}_{(N\text{-}M)\times M} \\ \mathbf{0}_{M\times(N\text{-}M)} & \mathbf{R}_{M\times M} \end{bmatrix}$$

$$\mathbf{S}'(t) = \mathbf{S}'_{N\times N} = \begin{bmatrix} \rho\mathbf{I}_{(N\text{-}M)\times(N\text{-}M)} & \\ & \mathbf{S}_{N\times M} \\ \mathbf{0}_{M\times(N\text{-}M)} & \end{bmatrix} \tag{39a-c}$$

$$\mathbf{r}'(t) = \mathbf{r}'_{N\times N} = [\rho \cdots \rho \;\; \mathbf{r}^T]^T$$

where ρ is a weighting constant chosen to be a large positive number. If **S**=0, **q**=0 and **r**=0, then Equation (37) reduces to the following equation with the simple penalty function:

$$L' = L + \rho \int_0^T \left[\hat{\mathbf{u}}^T(t)\, \hat{\mathbf{u}}(t)\right] dt \tag{40}$$

By penalizing the artificial control vector, the magnitude and influence of the artificial control variables can be made small and the solution of the modified optimal control problem can approximate the solution of the original LQ problem.

Example

This example, adapted and modified from (Meirovitch, 1990, Example 6.3), considers a series arrangement of J masses and J springs. As shown in Figure 1, it represents a 2J order system with a single force input acting on the last mass, m_J. The displacement of mass m_j is denoted by q_j. The mass and stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_J \end{bmatrix} \tag{41}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & & & 0 \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & & \ddots & & & \\ & & & -k_{J-1} & k_{J-1}+k_J & -k_J \\ 0 & & & & & k_J \end{bmatrix} \tag{42}$$

The state equation of this system is given by Equation (4) with

$$\mathbf{x} = [x_1\ x_2 \ldots x_{2J}]^T = [q_1\ q_2 \ldots q_J\ \dot{q}_1\ \dot{q}_2 \ldots\ \dot{q}_J]^T \tag{43}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ & \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix}, \qquad \mathbf{B} = [0\ 0\ \ldots\ 0\ 1/m_J]^T \tag{44),(45}$$

The initial conditions are

$$\mathbf{x}(0) = [x_1(0)\ x_2(0)\ \ldots\ x_{2J}(0)]^T \tag{46}$$

where it is presumed

$$x_J(0) = 1 \qquad x_j(0) = 0 \qquad j = 1,2,\ldots,J-1,J+1,\ldots,2J \tag{47a-b}$$

implying that the last mass only has been displaced from rest.

The problem is to find the optimal control history, u(t), that minimizes the performance index

$$L = \int_0^{10} \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + u^2\right) dt \tag{48}$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \tag{49}$$

The integrand term $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ represents the sum of kinetic and potential energies of the system. The inclusion of the integrand term u^2 reflects the desire to minimize the force (as well as the total energy).

Using the values $m_j=10kg$ and $k_j=1N/m$ (j=1,2,...,J) for three different systems, J=3, 5, and 7, the optimal solutions were determined using a Riccati approach, a transition matrix approach, and the proposed Chebyshev-based approach. The Chebyshev-based approach assumed an eight term series (*i.e.*, initial condition plus seven Chebyshev-type terms), and the artificial control variable technique with $\rho=10^5$ was employed . All simulation results were obtained on a Macintosh II platform running "C" programs. The resulting values of the performance index and the execution time are summarized in Table 1.

For J=3 the response histories of the state variables x_3 and x_6 (the displacement and velocity of the last mass, respectively) and the control variable u obtained using the Chebyshev-based approach are compared, respectively, with the state and control variables of the transition matrix approach in Figures 2 and 3. The 8-term Chebyshev-based solution is close to the transition matrix solution. To verify that the artificial control variable technique is successful, the time histories of the artificial control variable $\hat{u}_1$ for both 8-term and 10-term series are plotted in Figure 4. The figure shows that the artificial control variable based on a 10-term series is smaller in magnitude (closer to zero) than the artificial control variable based on a 8-term series. However, for both cases the magnitudes are small and hence the influence of the artificial control variables on the system dynamics is negligible.

In summary, the example demonstrates the applicability of the Chebyshev-based approach to general linear systems (with fewer control variables than state variables) and verifies the computationally efficient nature of the solution procedure.

Conclusion

The aim of this paper has been to present a user-friendly and computationally efficient Chebyshev-based algorithm for solving linear quadratic optimal control problems. The approach enables the necessary condition of optimality to be written as a set of linear algebraic equations. This is a key reason underlying the computationally streamlined nature of the approach. Another advantage of the approach, especially important for time-invariant problems, is the availability of closed-form formulas for the integrals of shifted Chebyshev polynomial terms. These integrals are needed in establishing the linear algebraic equations representing the condition of optimality. Finally, an artificial control variable technique is promoted as a means to make the approach tractable for general linear systems. Simulation results demonstrate computational advantages of the proposed approach relative to a Riccati approach and a transition matrix approach.

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Table 1. Example Simulation Results

	J=3		J=5		J=7	
METHOD	L	Time (sec)	L	Time (sec)	L	Time (sec)
Riccati	7.6205	45.5	7.6204	296	7.6204	1030
Transition Matrix	7.6205	3.75	7.6204	13.2	7.6204	34.7
Chebyshev-Based (8 terms)	7.6819	2.73	7.6858	7.57	7.6858	16.4

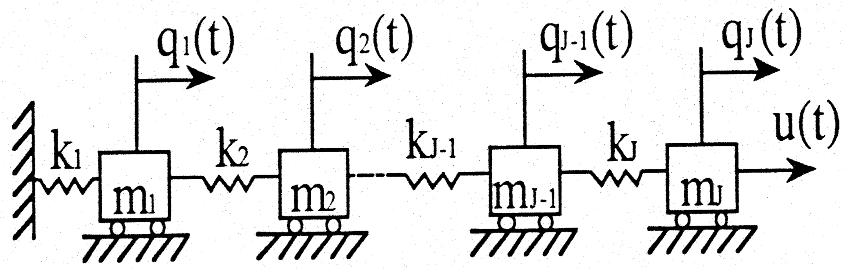

Figure 1. 2J Order System of Example

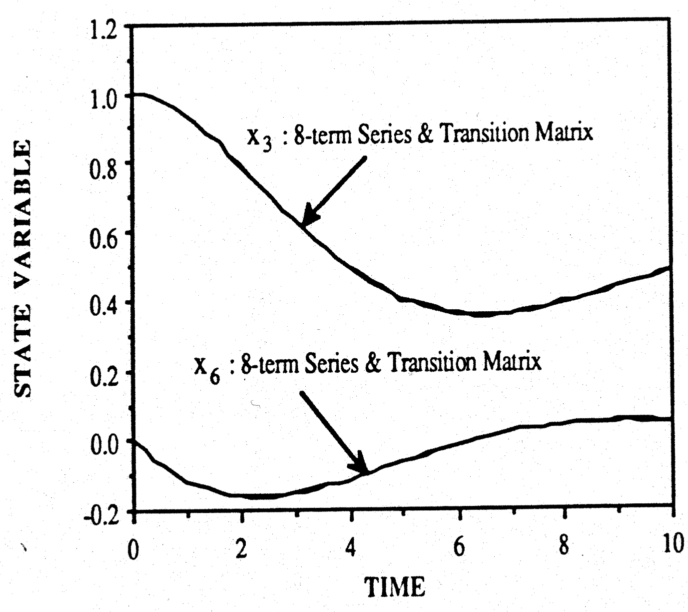

Figure 2. State Variable History of Example

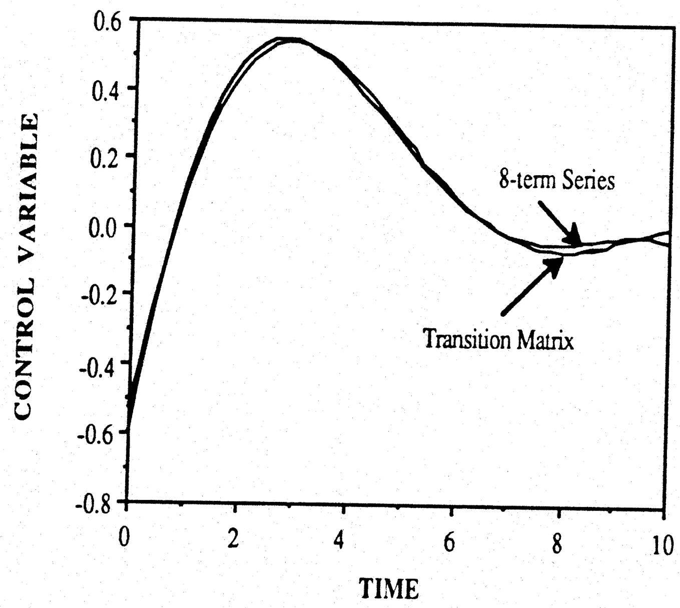

Figure 3. Control Variable History of Example

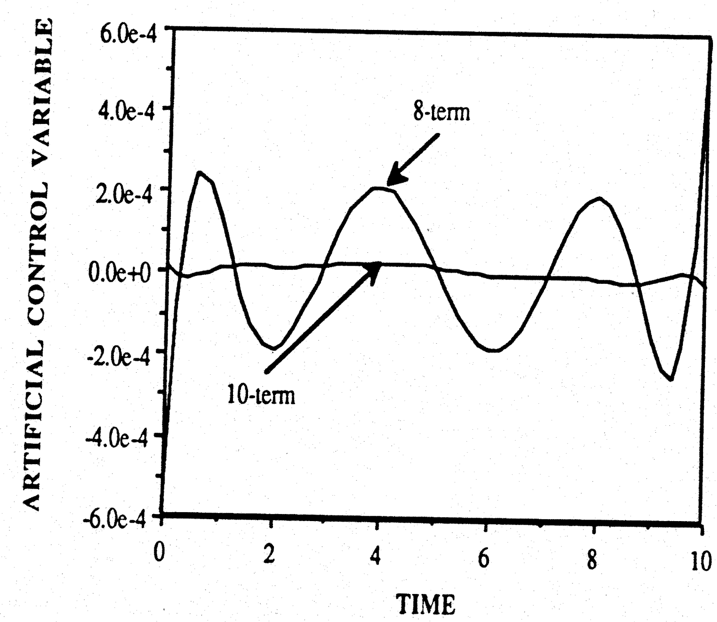

Figure 4. Artificial Control Variable ($\hat{u}_1$) History of Example