**O. Yaniv**
Faculty of Engineering
Department of Electrical Engineering Systems,
Tel Aviv University,
Tel Aviv 69978, Israel
e-mail: yaniv@eng.tau.ac.il

**M. Nagurka**
Department of Mechanical and Industrial Engineering,
Marquette University,
Milwaukee, WI 53201
e-mail: mark.nagurka@marquette.edu

# Automatic Loop Shaping of Structured Controllers Satisfying QFT Performance

*This paper presents a robust noniterative algorithm for the design of controllers of a given structure satisfying frequency-dependent sensitivity specifications. The method is well suited for automatic loop shaping, particularly in the context of Quantitative Feedback Theory (QFT), and offers several advantages, including (i) it can be applied to unstructured uncertain plants, be they stable, unstable or nonminimum phase, (ii) it can be used to design a satisfactory controller of a given structure for plants which are typically difficult to control, such as highly underdamped plants, and (iii) it is suited for design problems incorporating hard restrictions such as bounds on the high-frequency gain or damping of a notch filter. It is assumed that the designer has some idea of the controller structure appropriate for the loop shaping problem.*
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*Keywords: Automatic Loop Shaping, Robust Control, QFT, Linear Feedback Systems*

## 1 Introduction

Loop shaping is often a key step in the process of designing a controller. Although the skill can be gained easily, loop shaping is not simple, and many are not familiar with it. Moreover, in complicated applications it may be hard even for one skilled in the art. Automatic loop shaping (ALS) streamlines the design process and offers the possibility of finding a controller faster and better than that obtainable by a knowledgeable person.

ALS techniques can be split into two major categories: (i) those based on optimization algorithms, such as convex, nonconvex, and genetic algorithms (GAs), and (ii) those based on calculating a dense set of controllers from which the optimal solution can be chosen. The first approach affords more freedom in the sense that the controller structure is not fixed (although it is restricted in some way). In the second approach the controller structure is fixed and the design challenge is in selecting free parameters that offer optimal performance and satisfy constraints. Researchers have pursued both approaches.

An early effort towards the goal of achieving an ALS algorithm was reported by Gera and Horowitz [1]. They describe a semi-ALS process, i.e., an iterative approach in which the designer sequentially adds an element to the open-loop transfer function to force it to pass a given straight line in the Nichols chart. Being semi-automatic, the outcome may be a very high-order controller and the solution may be far from optimal. The technique was automated by Ballance and Gawthrop [2]. Chait et al. [3] offer an alternate ALS algorithm. They present a sufficient, constructive condition for converting the original problem into a convex optimization formulation. The main drawback of their technique is that the poles of the controller are fixed and are not part of the optimization process. Thompson and Nwokah [4] proposed a constrained, finite dimensional, nonlinear programming approach that starts from an assumed initial QFT controller. The technique was extended by Thompson [5] to QFT bounds for combined parametric and nonparametric as well as weighted control efforts. A GA for ALS was proposed by Garcia-Sanz and Guillen [6] with the advantage over previous work being that no initial controller is needed. However, it suffers from the drawbacks of GAs, such as being computationally demanding (GAs are inherently parallel algorithms and therefore best suited to multiple-processor machines), not guaranteeing a global optimum without unlimited computational effort, and being user intensive (knowing what one wants helps to setup the problem in a way that increases the chance of getting a better solution).

ALS algorithms falling into the second category have been pursued by Zolotas and Halikias [7], Besson and Shenton [8], and others. Zolotas and Halikias [7] describe an ALS method for proportional-integral-derivative (PID) controllers based on searching over a dense set of controllers. Their technique is efficient for two parameter controllers and a small number of bounds. Fransson et al. [9] adopted a nonconvex optimization method to design PID controllers for QFT-type problems, while considering the tradeoffs among low, mid, and high frequencies specifications.

This paper describes an approach that also fits within the second ALS category. It is based on a noniterative solution for the design of a two-parameter controller. As will be shown, it can be extended to design controllers of more than two parameters by cycling through free parameters two-at-a-time. In practice, it means solving for the optimal controller with two free parameters while fixing all other parameters, and then repeating this process over a reasonable range of values of the fixed parameters. As opposed to the existing optimization techniques, the approach proposed here is based on a set of closed form solutions, giving a set of controllers. Moreover, it can be used for the design of controllers with notch filters and can handle gain uncertainty without additional computational burden.

The paper is organized as follows: The problem is defined in Sec. 2 and the design method for two parameter controllers is given in Sec. 3. An extension for the design of controllers having more than two parameters is provided in Sec. 4. Section 5 addresses the design of two degree-of-freedom QFT problems.

## 2 The Loop Shaping Problems

The QFT loop shaping problem for single-input single-output (SISO) or multi-input single-output (MISO) systems or the sequential QFT loop shaping problem for multi-input multi-output (MIMO) systems can be described as follows: Find a stabilizing

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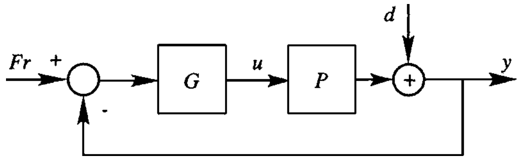


**Fig. 1 A MISO feedback system**

linear time-invariant (LTI) controller $G(s)$ such that the feedback system whose open-loop transfer function, $L(s)=P_0(s)G(s)$, satisfies

$$L(j\omega)=P_0(j\omega)G(j\omega)\in R(\omega), \quad \forall\ \omega\geqslant 0$$

where $P_0(s)$ denotes a single plant and $R(\omega)$, for any frequency $\omega$, denotes a set in the complex plane. Moreover, for a given controller structure, an optimal controller can be defined in any way. In QFT the practice is to take the controller whose high-frequency-gain (HFG) is minimum. (The HFG of $G(s)$ is $k=\lim_{s\to\infty}G(s)s^e$ where $e$ is the excess of poles over zeros of $G$.) It is required to design a controller whose HFG is as close to the optimal controller as possible.

For MISO QFT problems, the boundary of the set $R(\omega)$ can for some specifications be given by the QFT bounds (or Horowitz bounds [10]) of the following problem. Consider the feedback system depicted schematically in Fig. 1 and described by the equations,

$$y=Pu+d$$

$$u=G(Fr-y), \tag{1}$$

where $P$ is an LTI plant known to belong to a set $\{P\}$, $G$ is an LTI controller, $r=0$ and $d$ is a disturbance which may depend on $P\in\{P\}$. The problem is to design an LTI controller, $G(s)$, such that for any $P\in\{P\}$, the system (1) is stable and for a given function, $M(\omega,P)$, the transfer function $y(s)/d(s)$ is bounded by

$$\left|\frac{1}{1+P(j\omega)G(j\omega)}\right|\leqslant M(\omega,P), \quad \forall\ \omega\geqslant 0 \quad \text{and} \quad P\in\{P\}. \tag{2}$$

$R(\omega)$ is the set in the complex plane such that $P_0(j\omega)G(j\omega)\in R(\omega)$, if and only if, (2) is satisfied for $\omega$ (where $P_0$ is a chosen member of $\{P\}$ and $R(\omega)$ depends on $P_0$). Note that since $M$ can be a function of the plant, Eq. (2) can represent disturbances at the plant input and disturbances as a function of the plant.

## 3 Automatic Loop Shaping for a Two-parameter Controller

Let $P_1(s)$, $P_2(s)$ be a pair of plants known to belong to a set of pairs $\{(P_1(s),P_2(s))\}$ and define an open-loop transfer function $L(s)$ for the pair $P_1(s)$, $P_2(s)$ by

$$L(s)=a(P_1(s)+bP_2(s)) \tag{3}$$

where $a$ and $b$ are scalars. The design problem is to find the $a$, $b$ pairs such that

$$\left|\frac{1}{1+L(j\omega)}\right|\leqslant M(P_1(j\omega),P_2(j\omega),\omega);$$

$$\forall\ \omega\geqslant 0 \quad \text{for all pairs} \quad P_1,P_2\in\{(P_1,P_2)\}. \tag{4}$$

In order to identify the open-loop transfer function of the system of Fig. 1, $L(s)=PG$ is written as $L(s)=aP_1(s)(1+bP_2(s)/P_1(s))$. Thus, choosing $P_1=P$ and $P_2/P_1=s$ gives a PD controller, $G=a(1+bs)$; choosing $P_1=P/s$ and $P_2/P_1=1$ gives the PI controller, $G=a(1/s+b)$. A PD controller augmented with a low-pass filter, $G=a(1+bs)/(1+s/p)$, is the result of choosing $P_1=P/(1+s/p)$ and $P_2/P_1=s$. These and more structures will be used later to design controllers of more than two parameters.

The $(a,b)$ pairs satisfying inequality (4) form a set in the two-dimensional plane. An analytic algorithm for calculating its boundary was presented by Yaniv and Nagurka [11] for a PI controller and constant $M$ and in Ref. [12] for the open-loop case having the form of Eq. (3) with constant $M$. The algorithm for $M(\omega)$ is developed next.

Substituting (3) into (4) gives (where the dependence on $\omega$ is suppressed for convenience)

$$U+a(U_1+bU_2)+a^2(V_1+bV_2+b^2V_3)\geqslant 0, \quad \forall\ \omega\geqslant 0 \tag{5}$$

where

$$U=1-1/M^2, \quad U_1=2\cdot\mathrm{Real}(P_1), \quad U_2=2\cdot\mathrm{Real}(P_2),$$

$$V_1=|P_1|^2, \quad V_2=2\cdot\mathrm{Real}(P_1P_2^*), \quad V_3=|P_2|^2.$$

For an $(a,b)$ pair which is on the boundary of the allowed $(a,b)$ pairs, there exists $\omega$ and a pair $P_1,P_2\in\{(P_1,P_2)\}$ such that (5) is an equality. (Otherwise, $(a,b)$ can only be an internal point of the set solving the inequality (5).) At that particular $\omega$, the left-hand side of (5) is zero and minimum, and thus its derivative with respect to $\omega$ is zero. (The latter observation was also used in Ref. [13] to design a controller for maximum $a$.) Thus, at that particular $\omega$ (dot denotes derivative with respect to $\omega$)

$$\dot{U}+a(\dot{U}_1+b\dot{U}_2)+a^2(\dot{V}_1+b\dot{V}_2+b^2\dot{V}_3)=0. \tag{6}$$

Solving the equality of (5) and (6) for $a$ gives

$$a=\frac{\dot{U}U_1-U\dot{U}_1-(-\dot{U}U_2+U\dot{U}_2)b}{\dot{U}V_1-U\dot{V}_1+(\dot{U}V_2-U\dot{V}_2)b+(\dot{U}V_3-U\dot{V}_3)b^2}$$

$$=\frac{A+Bb}{C+Db+Eb^2}. \tag{7}$$

Substituting (7) into the equality of (5) gives a fourth-order polynomial equation for $b$

$$x_4b^4+x_3b^3+x_2b^2+x_1b+x_0=0 \tag{8}$$

whose coefficients are the following functions of $\omega$

$$x_4=UE^2+B^2V_3-BU_2E$$

$$x_3=(-U_2E+2BV_3)A-BU_1E+2UDE+B^2V_2-BU_2D$$

$$x_2=A^2V_3+(2BV_2-U_2D-U_1E)A+B^2V_1-BU_1D+UD^2+2UCE-BU_2C$$

$$x_1=A^2V_2+(-U_1D-U_2C+2BV_1)A+2UCD-BU_1C$$

$$x_0=-AU_1C+UC^2+A^2V_1.$$

The boundary of the allowed $(a,b)$ region can be calculated using the following procedure.

(1) Choose a pair $P_1,P_2\in\{(P_1,P_2)\}$.
(2) Choose $\omega$ and solve (8) for $b$. Noting that $b$ has four solutions (for a given $\omega$), pick the real solution (s).
(3) For each $b$ found in (2), solve inequality (4) to find all $a$, $b$ pairs for which the resulting closed-loop system is stable and (4) is satisfied for all $P_1,P_2\in\{(P_1,P_1)\}$. The result for each $b$ is one or more intervals since inequality (4) is a set of quadratic inequalities involving $a$.
(4) Repeat the above three steps over a range of frequencies, $\{\omega\}$, and all $P_1,P_2\in\{(P_1,P_2)\}$. This will give pairs,

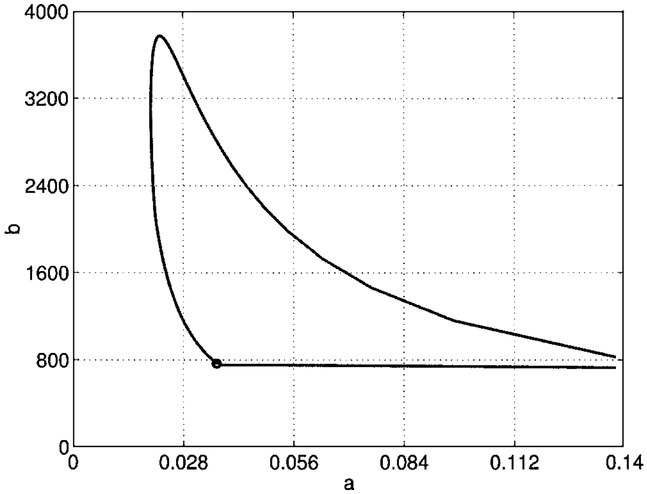


**Fig. 2 Boundary of all ($a,b$) pairs satisfying inequality (4) and (5) for example (1). The $a$, $b$ pair for which $ab$ is minimum is marked by (∘).**

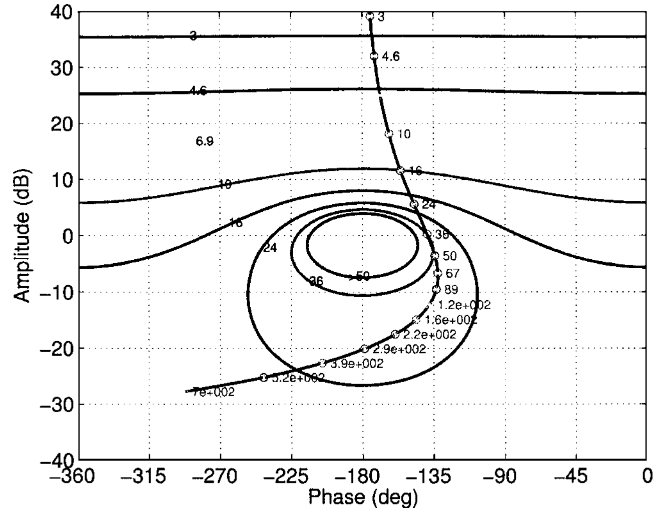


**Fig. 3 Several QFT bounds satisfying inequality (4) and the open-loop transfer function $L(s)$ for the pair $a$, $b$ for which $ab$ is minimum**

$(a(\omega),b(\omega))$, for each $\omega\in\{\omega\}$ having a solution in steps (2) and (3), which form the boundary of the allowed $(a,b)$'s.

Some remarks regarding the topology of the $(a,b)$ region can be found in Ref. [12].

A QFT optimal controller is a controller for which the HFG is minimum (see Horowitz and Sidi [14]). For $C(s)=a(1+bs)$ the optimal controller is the one for which $ab$ is minimum. The following lemma states that this optimal controller lies on the boundary of the $(a,b)$ region or is as close to the boundary for the accuracy associated with the frequency grid searched and plant grid searched, given the assumption that $P_1$ and $P_2$ are uncertain. It is therefore sufficient to calculate the boundary of the $(a,b)$ region to find the optimal controller.

**Lemma 3.1** *Let $(a,b)$ denote the set of all controllers solving the problem defined by equations* (3) *and* (4). *Suppose this set is not empty and denote by d the infimum of all the multiplications ab's within the allowed $(a,b)$'s. Then there exists an $a_0$, $b_0$ pair on the boundary of the allowed $(a,b)$'s such that $a_0b_0=d$.*

*Proof.* Denote by $a_1$, $b_1$ a pair for which $a_1b_1$ is infimum over all allowed $(a,b)$'s (such finite infimum exists because closed loop stability requires lower bound on $a$ and $b$). If $a_1$, $b_1$ is not on the boundary, then (5) is an inequality, thus, there exists $a_0=a_1-\epsilon$ such that for $a_0$, $b_1$ (5) is valid and the system is stable, which contradicts the assumption that $a_1b_1$ is infimum. Therefore, $a_1$, $b_1$ must be on the boundary. ■

**Example 1: Design of a PD Controller.** The plant pair $P_1$, $P_2$ as defined in Eq. (3) and the sensitivity upper bound, $M$, are

$$P_1=\frac{1}{s^2}e^{-sT},\quad T=0.005\ \text{s},$$

$$P_2=sP_1,$$

$$M=\left|\frac{2s^3}{(s+10)^2(s+30)}\right|,\quad s=j\omega.$$

The open-loop transfer function is $L=P_1G$ where $G=a(1+bs)$ is a PD controller. The boundary of the $(a,b)$'s is given in Fig. 2, calculated for 300 frequencies, between $\omega=1$ and 700 rad/s. In this and the following examples, frequencies are chosen such that the plant does not change by more than 1 deg and 0.2 dB).

From inequality (5) for each frequency, $\omega$, the boundary of the domain where $L(j\omega)$ must lie can be calculated. It is possible to draw these boundaries, known as QFT bounds. Figure 3 shows several such boundaries and the open-loop transfer function $L(s)=P(s)G(s)$, where

$$G(s)=a(1+bs)=820(1+0.0348s)$$

is the controller for which the HFG is minimum.

## 4 Automatic Loop Shaping of More Than Two-Parameter Controllers

An open-loop transfer function of the two-parameter controller can be written in the form

$$L(s)=aP_1(s)(1+bP_2(s)/P_1(s)).$$

Let $P$ denote a plant and choose $P_1=PH$ and $P_2/P_1=W$, which gives the open-loop transfer function $L=PG$ where the controller is $G=aH(1+bW)$. This observation is the basis for extending the method for the design of a controller of arbitrary structure. For example, consider the design of a lead or lag controller

$$G(s)=\frac{a(1+bs)}{1+s/c}.$$

The following algorithm can be used to find suitable $(a,b,c)$ triplets and the triplet whose HFG is the lowest.

(1) Fix the pole $c$. This will reduce the problem of searching the boundary of $(a,b)$ to a two-parameter controller given in Sec. 3.
(2) Find the $(a,b)$ boundary and save the best $a$, $b$ pairs. If the criterion is lowest HFG, it is the pair for which $ab$ is minimum.
(3) Repeat steps (l) and (2) for a reasonable range of $c$ values. (A guideline for how to choose the range of $c$ values is given in example (2).)
(4) From all chosen controllers choose the best one one. If the criterion is lowest HFG, it is the one for which $abc$ is minimum.

**Remark 4.1** *The extension of lemma 3.1 to a three parameter controller is straightforward. Since for each searched c the near-optimal $a_c$, $b_c$ pair is known, it is possible to directly select a global near-optimum solution from all calculated triplets $\{a_c,b_c,c\}$. The denser the grid of $c$, the closer the solution will be to the global optimum. Moreover, for many engineering systems it*

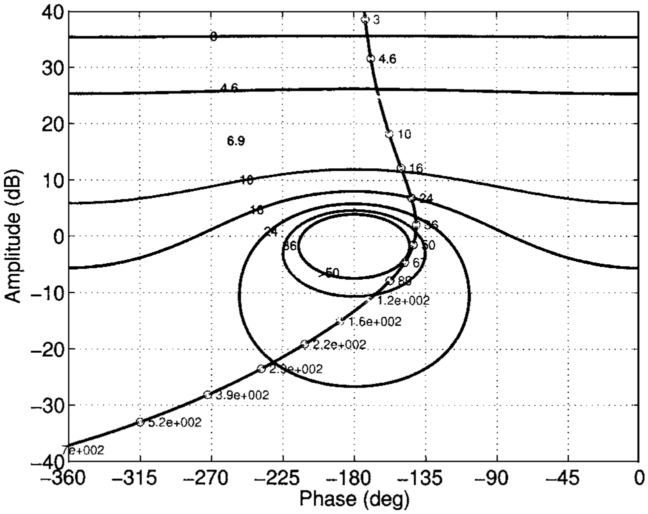


**Fig. 4 Several QFT bounds satisfying inequality (4) at different frequencies and the open-loop transfer function for the optimal triplet *a*, *b*, *c* for which *abc* is minimum**

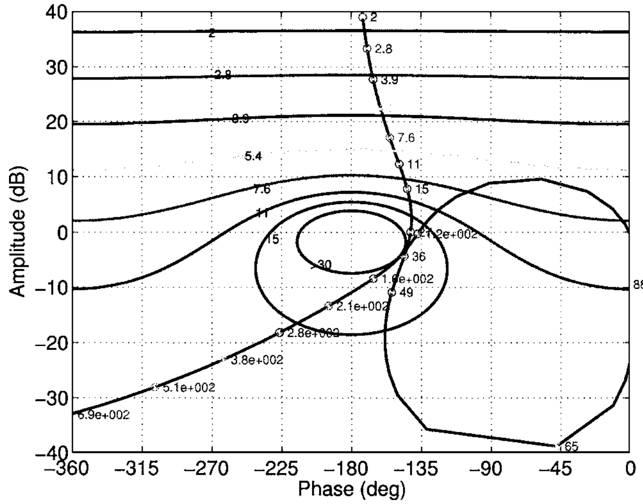


**Fig. 5 Several QFT bounds satisfying inequality (4) and the open-loop transfer function for which the controller HFG is minimum (nominal plant is the one with the largest resonance frequency, here 105 rad/s)**

*is straightforward to choose a frequency grid such that the boundary of all $a_c$, $b_c$ pairs is calculated for points that differ by less than a percent, and thus the optimal solution is close to the one taken from the calculated solutions.*

**4.1 Example 2: Design of a Gain and Lead or Lag Controller.** The plant and sensitivity function are the same as of those of Example 1. The frequency where the optimal solution of Example 1 crosses −180 deg below the 0 dB line is 290 rad/s (that is, $c \to \infty$ in this example). For the pole $c$ to be effective, it should therefore contribute at least 5 deg phase lag at 290 rad/s. Here, the search for the pole $c$ was started at 2900 and its value was decreased by 10% each iteration until it was sufficiently small that no solution existed. The optimal controller, found to be

$$G(s) = \frac{768(1+0.0535s)}{1+s/155},$$

is shown in Fig. 4, together with several QFT bounds. In comparison to the PD controller of example (1), it shows the same performance with much lower control effort (i.e., the open-loop transfer function above 0 dB is the same, while at high frequencies it is much lower).

The main question of this search-type algorithm relates to the computational burden. For this example, the algorithm took 0.17 s. on a 1.6 GHz laptop using MATLAB® v.6.5 where the routine which solves the fourth-order Eq. (8) was written efficiently in *C*.

**4.2 Example 3: Design of a Gain Notch and Lead or Lag Controller.** Many mechanical servo applications can be modeled as a load on a motor shaft. A classical model of such a system includes two integrators, a resonance, an anti-resonance whose frequency is lower than the resonant frequency, and a pure delay which may resemble a low-pass filter or a ZOH, that is,

$$P(s) = \frac{1}{s^2} \cdot \frac{s^2 + 2d_1\omega_1 s + \omega_1^2}{s^2 + 2d_2\omega_2 s + \omega_2^2} \cdot e^{-sT}.$$

A suitable controller for many applications of this type includes a lead element, a notch filter and a low-pass filter.

The following example investigates a controller with a lead element and a notch filter. The addition of a low-pass filter could readily be included. The controller model is

$$G(s) = \left[\frac{a(1+bs)}{1+s/c}\right]\left[\frac{1+2d_3\omega_3/s+\omega_3^2/s^2}{1+2d_4\omega_3/s+\omega_3^2/s^2}\right].$$

Consider the problem of finding the lowest HFG controller, $G(s)$, that stabilizes the uncertain plant, $P(s)$, whose parameters are: $d_1=0.01$, $d_2=0.03$, the resonance-antiresonance pair is uncertain being one of the following: $\omega_1=50$ rad/s, $\omega_2=90$ rad/s or $\omega_1=55$ rad/s, $\omega_2=95$ rad/s, or $\omega_1=60$ rad/s, $\omega_2=100$ rad/s, or $\omega_1=65$ rad/s, $\omega_2=105$ rad/s and the delay $T$ is 0.005 s or lower. The closed-loop sensitivity performance is defined by

$$M(\omega) = \left|\frac{2s^3}{(s+10)^2(s+10)}\right|, \quad s=j\omega.$$

The following design algorithm was applied.

(1) Since a suitable notch filter should notch around the resonance, choose one of the following filters.

(a) $d_4=0.5$ and $d_3$ a value between 0.07 and 0.3, with seven equally spaced values in logarithmic scale.

(b) A value of $\omega_3$ between the lowest resonance, 90 rad/s and 1.5 times the highest resonance, 105 rad/s, with eight values equally spaced in logarithmic scale.

(2) The pole, $c$, of the low-pass filter is chosen. Starting with a very large value of $c$, find where the open-loop transfer function crosses −180 deg below the 0 dB line, say at $c_\infty$. Search from a maximum value of $c=10c_\infty$ and decrease by 10% in each iteration until no solution exists.

(3) Search for the boundary of the $(a,b)$ domain and from it select the controller considered the optimal one.

(4) Repeat the previous steps for all chosen notch filters and low-pass filters. Save the optimal solution(s) for each iteration.

(5) From all saved optimal solutions choose a global optimal one.

Implementing this algorithm gave the following controller

$$G(s) = \left[\frac{918(1+0.0774s)}{1+s/136}\right]\left[\frac{1+2\cdot 0.224\cdot 95/s+95^2/s^2}{1+2\cdot 0.5\cdot 95/s+95^2/s^2}\right].$$

Figure 5 presents the open-loop transfer function where the nominal plant $P_0$ is the one with the lowest resonance frequency (90 rad/s). It also includes several QFT bounds. The computational cost (using a 1.6 GHz laptop and Matlab) associated with this four element controller was 35 s.

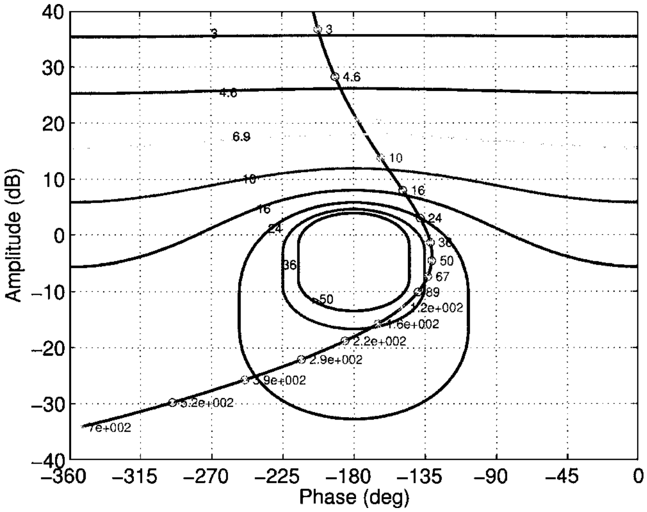


**Fig. 6 Several QFT bounds satisfying inequality (4) and the system's open-loop transfer function (for $k$=1) for which the controller HFG is minimum**

**4.3 Example 4: Design of a PID Controller With Filtered *D*.** More than 90% of industrial feedback controllers are PI, PD, or PID with a low-pass filter on the $D$-term [15,16], that is, the controller structure is of the following form:

$$G(s) = \frac{k_I}{s} + k_P + \frac{k_D s}{1 + s/c},$$

with the four design parameters $(k_I, k_P, k_D, c)$. For this example, the uncertain plant and performance are

$$P = \frac{k}{s^2 + s} e^{-sT}, \quad T = 0.005 \text{ s}, \quad \text{gain uncertainty } k \in [1,2]$$

$$M = \left| \frac{2s^3}{(s+10)^2(s+30)} \right|, \quad s = j\omega.$$

The design process is as follows. First, a PID controller is designed and the frequency where the optimal solution crosses $-180$ deg below 0 dB is calculated. (This occurs at 290 rad/s.) The expected pole on the $D$ term should therefore be around that value. A search for the pole in the frequency range [70, 2900] was conducted. Searching over forty $c$ values equally spaced logarithmically suffices since successive $c$ values then differ by less than 10%. The optimal controller was found to be

$$G(s) = \frac{1530}{s} + 506 + \frac{27.2}{1 + s/387}$$

The open-loop transfer function for the plant where $k$=1 is shown in Fig. 6 together with some QFT bounds. This figure verifies graphically both stability and satisfaction of the specifications.

An important question is what conditions must exist for a successful application of the algorithm proposed here. A key condition is that there is a fixed controller $G_0(s)$ such that $(a + bs)G_0(s)$ is a solution to the problem for some $a$, $b$ values, or, more generally, there are fixed controllers, $G_1$ and $G_2$, such that $aG_1 + bG_2$ is a solution to the problem for some $a$, $b$ values. Furthermore, it is assumed that the designer has some idea of the controller structure appropriate for the problem.

## 5 The Design of Two DOF Systems by Reduction to One DOF Systems

Consider the feedback system depicted schematically in Fig. 7 and described by the equations

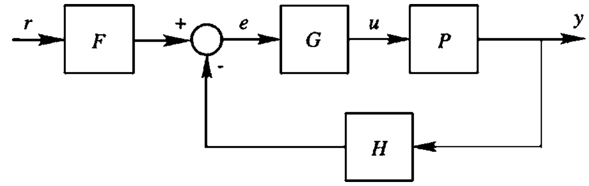


**Fig. 7 A SISO two degree-of-freedom feedback system**

$$y = Pu$$

$$u = Fr - GHy,$$

where $P$ is a LTI plant known to belong to a set $\{P\}$, $G$ is an LTI controller, $r$ is the reference input, $F$ is the prefilter, and $H$ is the sensor. The QFT two degree-of-freedom problem can be phrased in two ways.

(1) Given a function $A(\omega)$, design $F$ and $G$ such that

$$\left| \frac{P(j\omega)G(j\omega)F(j\omega)}{1 + P(j\omega)G(j\omega)} - F(j\omega) \right| \leq A(\omega), \quad \text{for all} \quad P \in \{P\}.$$

(2) Given two functions $A(\omega)$ and $B(\omega)$, design $F$ and $G$ such that

$$A(\omega) \leq \left| \frac{P(j\omega)G(j\omega)F(j\omega)}{1 + P(j\omega)G(j\omega)} \right| \leq B(\omega), \quad \text{for all} \quad P \in \{P\}.$$

The first problem, assuming $F$ is given, is analogous to the ALS problem defined in Sec. 3 because it is equivalent to

$$\left| \frac{1}{1 + P(j\omega)G(j\omega)} \right) \leq \frac{A(\omega)}{|F(j\omega|}, \quad \text{for all} \quad P \in \{P\}.$$

The second problem, where $F$ is assumed known, is also analogous to this problem based on the following lemma.

**Lemma 5.1** *Let $M \neq 1$ be a constant. A complex number $L$ satisfies the inequality*

$$\left| \frac{L}{1 + L} \right| \leq M$$

*if and only if it satisfies the inequality*

$$\left| \frac{1}{1 + L/M_0} \right| \leq M, \quad \text{if} \quad M > 1$$

$$\left| \frac{1}{1 + L/M_0} \right| \geq M, \quad \text{if} \quad M < 1; \quad M_0 \frac{M^2}{M^2 - 1}.$$

*Proof.* Substitute $L$=$x$+$jy$ in both inequalities and after manipulation it can be shown that both are the same inequality. ■

Note that a bound for $M$=1 can always be calculated by replacing it by a value very close to 1. Based on lemma 5.1 a design process for the second two degree-of-freedom problem, suited for the design of a two-parameter controller, is

(1) Choose the prefilter $F$, the problem will then be analogous to designing $G$ such that

$$M_1 = \frac{A(\omega)}{|F(j\omega)|} \leq \left| \frac{P(j\omega)G(j\omega)}{1 + P(j\omega)G(j\omega)} \right| \leq \frac{B(\omega)}{|F(j\omega)|} = M_2,$$
$$\text{for all} \quad P \in \{P\}. \tag{9}$$

(2) Find the boundary of the $(a,b)$'s solving the right side of inequality (9)

$$\left| \frac{1}{1 + P(j\omega)G(j\omega)/M_0} \right| \leqslant M_1, \quad \text{for all} \quad P \in \{P\}$$

where $M_0 = M_1^2/(M_1^2 - 1)$, change the inequality if $M_1 < 1$ according to lemma 5.1.

(3) Find the boundary of the $(a,b)$'s solving the left side of inequality (9)

$$\left| \frac{1}{1 + P(j\omega)G(j\omega)/M_0} \right| \geqslant M_2, \quad \text{for all} \quad P \in \{P\}$$

where $M_0 = M_2^2/(M_2^2 - 1)$, change the inequality if $M_1 < 2$ according to lemma 5.1.

(4) The intersection of the closure of the latter two domains is all $(a,b)$ pairs being sought. Pick an optimal $a$, $b$ pair as a solution.

The extension to controllers involving more than two parameters is as described in Sec. 4.

## 6 Conclusions

The paper presents a noniterative based algorithm to design structured controllers satisfying frequency-dependent sensitivity specifications. Assuming that the designer has some idea of the controller structure and a suitable range of its parameters, the proposed approach can be used to find the boundary of the set of possible solutions. Internal points are of no interest if the QFT optimal criterion is used.

Being noniterative, the algorithm is efficient and fast, and offers several additional attractive properties. It can be applied to unstructured uncertain plants, be they stable, unstable or nonminimum phase. It can be used to design a near optimal controller of a given structure for plants which are typically difficult to control, such as highly underdamped plants. The algorithm can be used to sort controller designs based on a given criterion, such as the one with the lowest high-frequency gain. The technique can be extended to incorporate notch filters and low-pass filters into the controller. In addition, hard restrictions can be included, such as bounding the damping factor of a notch filter or adding a low-pass filter with a given cutoff frequency in order to limit the sensor noise effect.

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