Fourier-Based Optimal Control Approach for Structural Systems

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This paper considers the optimal control of structural systems with quadratic performance indices. The proposed approach approximates each configuration variable of a structural model by the sum of a fifth-order polynomial and a finite-term Fourier-type series. In contrast to standard linear optimal control approaches, the method adopted here is a near optimal approach in which the necessary condition of optimality is derived as a system of linear algebraic equations. These equations can be solved directly by a method such as Gaussian elimination. The proposed approach is computationally efficient and can be applied to structural systems of high dimension and to systems with linear boundary constraints.

Introduction

THE problem of determining the optimal control of a linear dynamic system with a quadratic performance index is usually solved by a variational approach. Mathematically, this linear quadratic (LQ) problem can be posed as a two-point boundary-value problem (TPBVP). The initial conditions are specified for the state equations and the terminal conditions are specified for the costate equations, with the set of state and costate equations often called the Hamiltonian system. Standard routines for solving linear boundary-value problems are generally inefficient in solving such a TPBVP.[1] More efficient methods specifically designed to solve the LQ problem are available. These can be classified as closed-loop and open-loop approaches.

The most widespread closed-loop approach is based on the solution of a matrix differential Riccati equation. Various algorithms have been proposed to solve the Riccati equation.[2] In contrast, the open-loop approach converts the TPBVP into an initial value problem by evaluating the exponential of the Hamiltonian matrix (i.e., the transition matrix of the Hamiltonian system). An example of a structural application of this open-loop approach is described by Turner and Chun.[3] A detailed discussion of the closed-loop Riccati equation approach and the open-loop transition-matrix approach can be found in Speyer.[4]

The Riccati-based approach is preferred for physical implementation due to the inherent advantages of closed-loop configurations. However, it is computationally more costly than the transition-matrix approach in solving time-invariant LQ problems. In particular, to generate the optimal response of an Nth-order system, the closed-loop approach requires the solution of $N(N+3)/2$ first-order differential equations [N state equations and $N(N+1)/2$ Riccati equations]. In contrast, the open-loop approach requires the integration of $2N$ first-order differential equations (N state equations and N costate equations). As a result, efficient software simulation tools for solving time-invariant LQ problems are typically based on the open-loop transition-matrix approach. For solving time-varying LQ problems, on the other hand, the Riccati-based approach is usually computationally more efficient. The time to evaluate the exponential of the time-varying Hamiltonian matrix is often greater than the time to integrate the Riccati equation.

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As an alternative to the preceding methods, Nagurka and Yen[5] proposed a Fourier-based approach to generate near-optimal trajectories of general dynamic systems. The basic idea of the approach is to represent the time history of each generalized coordinate by the sum of an auxiliary polynomial and a finite-term Fourier-type series. The free variables, such as the free coefficients of the polynomial and the Fourier-type series, are adjusted by a nonlinear programming method to minimize a performance index. The effectiveness of this technique has been demonstrated via simulation studies.[5]

This research specializes this Fourier-based approach to linear structural systems with quadratic performance indices. The method exploits the linearity of the system model and the quadratic nature of the performance index to guarantee identification of a global minimum, while being computationally efficient. As demonstrated by simulation results, the method offers computational advantages relative to previous closed and open-loop methods and, as such, promises to be the basis for efficient software for the design of optimal quadratic controllers for structural systems.

Methodology

The behavior of a controlled linear structure is governed by the equation of motion:

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = Bu(t)$$

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0 \tag{1}$$

where x is an $N \times 1$ configuration vector (i.e., a column vector of N configuration variables), u is a $J \times 1$ control vector, M is an $N \times N$ positive definite mass matrix, C is an $N \times N$ positive semidefinite structural damping matrix, K is an $N \times N$ positive semidefinite stiffness matrix, and B is an $N \times J$ control influence matrix. In this paper, it is assumed that J is less than or equal to N, i.e., the number of control variables is less than or equal to the number of configuration variables. The derivation that follows and examples 1–4 consider the actively controlled case $J = N$, i.e., the configuration and control vectors have the same dimension, with B nonsingular. Example 5 addresses the case $J < N$ for structural systems that are not actively controlled.

The design goal is to find the optimal control $u(t)$ in the time interval $[0,T]$ such that the quadratic performance index L is

$$L = z^T(T)\, Hz(T) + \int_0^T (z^TQz + u^TRu)\, \mathrm{d}t \tag{2}$$

is minimized while satisfying the equation of motion [Eq. (1)]. In Eq. (2) z is a state variable vector defined as

$$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{3}$$

It is assumed that H and Q are real, nonnegative-definite symmetric matrices and R is a positive-definite symmetric matrix. In addition, it is assumed that Q can be partitioned as

$$Q = \begin{bmatrix} Q_a & \tfrac{1}{2}Q_c^T \\ \tfrac{1}{2}Q_c & Q_b \end{bmatrix} \tag{4}$$

The performance index can thus be rewritten as

$$L = L_1 + L_2 \tag{5}$$

where

$$L_1 = z^T(T)Hz(T) \tag{6}$$

$$L_2 = \int_0^T (x^TQ_ax + \dot{x}^TQ_b\dot{x} + \dot{x}^TQ_cx + u^TRu)\, \mathrm{d}t \tag{7}$$

where L_1 is the cost associated with the terminal configuration and its rate, and L_2 is the cost associated with the trajectory. It is assumed that the configuration and control vectors are not bounded, the terminal time T is fixed, and the terminal configuration $x(T)$ is free or linearly constrained.

The following subsection describes the underlying idea of Fourier-based optimal control. The subsequent subsection applies the Fourier-based approach to solve the unconstrainted LQ problem for an actively controlled structure. The solution of LQ problems with linear boundary constraints is addressed later.

Fourier-Based Approach

For a structural system, the optimal control can be converted into a mathematical programming problem by directly representing the configuration variable by a Fourier series. In particular, by assuming the optimal profile of the ith configuration variable $x_i(t)$ to be continuous in the interval $[0,T]$, its Fourier series will converge to $x_i(t)$ in $(0,t_f)$, i.e.,

$$x_i(t) = a'_{i0} + \sum_{k=1}^{\infty} \left(a'_{ik} \cos\frac{2k\pi t}{T} + b'_{ik} \sin\frac{2k\pi t}{T} \right) \tag{8}$$

In practice, only a finite number of terms of the Fourier series is taken. An appropriate mathematical programming algorithm can then be used to determine the optimal values of the corresponding coefficients, which will determine an admissible trajectory that minimizes the performance index. This direct application of the Fourier series, however, has the following disadvantages:

1) Convergence is guaranteed only in $(0,T)$ unless the optimal $x_i(t)$ has identical boundary values.[6] Since the trajectory is defined in $[0,T]$, the convergence should extend from $(0,T)$ to $[0,T]$.

2) Although the Fourier series converges to $x_i(t)$, the derivative of the Fourier series does not necessarily converge to the derivative of $x_i(t)$. Convergence of the first and second derivatives is necessary since these derivatives appear explicitly in the equation of motion [Eq. (1)].

3) The speed of convergence of the Fourier series, which depends on the optimal $x_i(t)$, can be quite slow.

One method of overcoming the preceding difficulties is to append to the series a linear function of time such that

$$x_i(t) = [x_i(T) - x_i(0)\, \frac{t}{T} + a''_{i0}$$

$$+ \sum_{k=1}^{\infty} \left(a''_{ik} \cos\frac{2k\pi t}{T} + b''_{ik} \sin\frac{2k\pi t}{T} \right) \tag{9}$$

which can be rewritten as

$$w_i(t) = a''_{i0} + \sum_{k=1}^{\infty} \left(a''_{ik} \cos\frac{2k\pi t}{T} + b''_{ik} \sin\frac{2k\pi t}{T} \right) \tag{10}$$

where

$$w_i(t) = x_i(t) - [x_i(T) - x_i(0)](t/T) \tag{11}$$

is a linear function of time with the property

$$w_i(0) = w_i(T) \tag{12}$$

As a result, $w_i(t)$ can be treated as a periodic function with period T, which implies that its Fourier series has the following properties[6]:

1) The convergence interval of the Fourier series of $w_i(t)$ extends from $(0,T)$ to $[0,T]$.

2) In the interval of $(0,T)$, the term-by-term differentiation of the Fourier series of $w_i(t)$ converges to the first derivative of $w_i(t)$.

3) The speed of convergence of the Fourier series of $w_i(t)$ is more rapid than the speed of convergence of the Fourier series of $x_i(t)$.

In summary, by adding a suitable linear function of time to $x_i(t)$, a new periodic function, $w_i(t)$, can be generated which has equal values at the boundaries of $[0,T]$. Consequently, convergence on the boundaries and term-by-term differentiation can be guaranteed, and the rate of convergence of the Fourier series can be improved.

It should be noted that based on the property of Eq. (12), the derivative of the Fourier series of $w_i(t)$ converges to $\dot{w}_i(t)$ in $(0,T)$. This convergence interval can be extended to $[0,T]$ if $\dot{w}_i(t)$ has the same values at the boundaries.[6] In general, this feature cannot be achieved by appending only a linear function to the Fourier series. However, by adding a suitable polynomial of time to $x_i(t)$, it is possible to make the function as well as several of its derivatives have equal values at the boundaries. For example, if

$$w_i(0)=w_i(T) \tag{13}$$

$$\dot{w}_i(0)=\dot{w}_i(T) \tag{14}$$

$$\ddot{w}_i(0)=\ddot{w}_i(T) \tag{15}$$

then it can be shown that over the interval $[0,T]$ the first and second derivatives of the Fourier series of $w_i(t)$ converge to $\dot{w}_i(t)$ and $\ddot{w}_i(t)$, respectively. Also, the rate of convergence of the Fourier coefficients of $w_i(t)$ becomes three orders faster than the rate of convergence of the Fourier coefficients of $x_i(t)$.[6]

Equations (13-15) can be satisfied by appending a third-order polynomial (without the constant term) to $x_i(t)$. However, in the approach proposed here each configuration variable is represented by the sum of a fifth-order polynomial and a finite-term Fourier-type series. By raising the order of the polynomial from three to five, the five coefficients of the polynomial can be adjusted to satisfy Eqs. (13-15) as well as the constraints imposed by the initial conditions of $x_i(t)$ and $\dot{x}_i(t)$. (There are five coefficients since the constant term is not included in the polynomial, but rather the Fourier-type series.)

In summary, the proposed approach is to approximate each configuration variable by the sum of an auxiliary polynomial and a finite-term Fourier-type series, i.e., for $i=1,\ldots,N$,

$$x_i(t)=\sum_{k=0}^{5} d_{ik}t^k+\sum_{k=1}^{K}\left(a_{ik}\cos\frac{2k\pi t}{T}+b_{ik}\sin\frac{2k\pi t}{T}\right) \tag{16}$$

Note that, for convenience, the constant term of the Fourier series has been included in the auxiliary polynomial. By using boundary-value equations for the configuration variable and its rates, the six coefficients of this fifth-order auxiliary polynomial can be expressed as functions of the boundary values of $x_i(t)$, $\dot{x}_i(t)$, and $\ddot{x}_i(t)$. Explicit expressions for these coefficients are given in Appendix A.

Using the results of Appendix A, Eq. (16) can be rearranged and presented in the form

$$x_i(t)=p_i+\rho_1\ddot{x}_{i0}+\rho_2 x_{iT}+\rho_3\dot{x}_{iT}+\rho_4\ddot{x}_{iT}+\sum_{k=1}^{K}\alpha_k a_{ik}+\sum_{k=1}^{K}\beta_k b_{ik} \tag{17}$$

where $x_{i0}=x_i(0)$, $x_{iT}=x_i(T)$, and similarly for the corresponding time derivatives, and where

$$p_i=x_{i0}+\dot{x}_{i0}t+(-10x_{i0}-6\dot{x}_{i0}T)\,\tau^3+(15x_{i0}+8\dot{x}_{i0}T)\,\tau^4+(-6x_{i0}-3\dot{x}_{i0}T)\,\tau^5 \tag{18}$$

$$\rho_1=T^2\,(0.5\tau^2-1.5\tau^3+1.5\tau^4-0.5\tau^5) \tag{19}$$

$$\rho_2=(10\tau^3-15\tau^4+6\tau^5) \tag{20}$$

$$\rho_3=T(-4\tau^3+7\tau^4-3\tau^5) \tag{21}$$

$$\rho_4=0.5T^2(\tau^3-2\tau^4+\tau^5) \tag{22}$$

$$\alpha_k=-1+\upsilon_k^2\,(0.5\tau^2-\tau^3+0.5\tau^4)+\cos(\upsilon_k\tau) \tag{23}$$

$$\beta_k=\upsilon_k(-\tau+10\tau^3-15\tau^4+6\tau^5)+\sin(\upsilon_k\tau) \tag{24}$$

with

$$\upsilon_k=2k\pi \tag{25}$$

and

$$\tau=t/T \tag{26}$$

Equation (17) can be written in compact form as

$$x_i=\rho^T y_i+p_i \tag{27}$$

where

$$\rho^T=[\rho_1\quad\rho_2\quad\rho_3\quad\rho_4\quad\alpha_1\quad\ldots\quad\alpha_K\quad\beta_1\quad\ldots\quad\beta_K] \tag{28}$$

$$y_i=[\ddot{x}_{i0}\quad x_{iT}\quad\dot{x}_{iT}\quad\ddot{x}_{iT}\quad a_{i1}\quad\ldots\quad a_{iK}\quad b_{i1}\quad\ldots\quad b_{iK}]^T \tag{29}$$

where ρ is a vector of known time functions used by the Fourier-based approach, and y_i is a vector of unknown parameters for the ith configuration variable. The length of both vectors is $m=4+2K$. Compared to Eq. (16), Eq. (27) decouples the unknown parameters from the known parameters of the configuration variable (in this case, the initial values, x_{i0} and $\dot{x}_{i0}$, which are embedded in p_i).

The configuration vector of the N-degree-of-freedom structure can be represented by

$$x=\bar{\rho}y+p \tag{30}$$

where

$$x=[x_1\quad x_2\quad\ldots\quad x_N]^T \tag{31}$$

$$y=[y_1^T\quad y_2^T\quad\ldots\quad y_N^T]^T \tag{32}$$

$$p=[p_1\quad p_2\quad\ldots\quad p_N]^T \tag{33}$$

$$\bar{\rho}=\begin{bmatrix}\rho^T & 0 & \cdot & \cdot & \cdot & 0\\ 0 & \rho^T & \cdot & & & \cdot\\ \cdot & & \cdot & & & \cdot\\ \cdot & & & \cdot & & \cdot\\ \cdot & & & & \cdot & 0\\ 0 & \cdot & \cdot & \cdot & 0 & \rho^T\end{bmatrix} \tag{34}$$

By direct differentiation, the configuration rate vectors can be expressed as

$$\dot{x}=\bar{\sigma}y+q \tag{35}$$

$$\ddot{x}=\bar{\gamma}y+r \tag{36}$$

where

$$\bar{\sigma}=\dot{\bar{\rho}},\qquad q=\dot{p} \tag{37}$$

$$\bar{\gamma}=\ddot{\bar{\rho}},\qquad r=\ddot{p} \tag{38}$$

Thus, using the Fourier-based approach the configuration variables x, $\dot{x}$, and $\ddot{x}$ can be written as known functions of y.

Actively Controlled Linear Quadratic Problems

This subsection implements the Fourier-based approach to solve the LQ problem for an actively controlled structural system with free terminal states. Equation (1) can be rewritten as

$$u = B^{-1}(M\ddot{x} + C\dot{x} + Kx) \tag{39}$$

since B has been assumed to be square and nonsingular. By substituting Eqs. (30), (35), and (36) into Eq. (39), the control vector can be written as a function of y:

$$u = B^{-1}(M\bar{\gamma} + C\bar{\sigma} + K\bar{\rho})y + B^{-1}(Mr + Cq + Kp) \tag{40}$$

Ultimately, the interest is to express the performance index as a function of y. Toward this end, the terminal state is written as a linear transformation of y, i.e.,

$$z(T) = \Theta y \tag{41}$$

where Θ is a $2N \times mN$ transformation matrix with elements 1 and 0, specified according to

$$\theta_{ij} = \begin{cases} 1, \; j = (i-1)m + 2 & \text{for} \quad i = 1,\dots,N \\ \quad j = (i-N-1)m + 3 & \text{for} \quad i = N+1,\dots,2N \\ 0, \text{ otherwise} \end{cases} \tag{42}$$

By substituting Eq. (41) into Eq. (6), the cost L_1 is

$$L_1 = y^T(\Theta^T H \Theta)y \tag{43}$$

By substituting Eqs. (30), (35), (36), and (40) into Eq. (7), the cost L_2 is

$$L_2 = y^T\Delta y + y^T\Gamma + \mu \tag{44}$$

where Δ is a matrix, Γ is a vector, and μ is a scalar:

$$\Delta = \int_0^T (\bar{\gamma}^T F_1 \bar{\gamma} + \bar{\sigma}^T F_2 \bar{\sigma} + \bar{\rho}^T F_3 \bar{\rho} + \bar{\gamma}^T F_4 \bar{\sigma} + \bar{\gamma}^T F_5 \bar{\rho} + \bar{\sigma}^T F_6 \bar{\rho}) \mathrm{d}t \tag{45}$$

$$\Gamma = \int_0^T (2\bar{\gamma}^T F_1 r + \bar{\sigma}^T (F_2 + F_2^T) q + \bar{\rho}^T (F_3 + F_3) p + \bar{\gamma}^T F_4 q + \bar{\gamma}^T F_5 p + \bar{\sigma}^T F_6 p + \bar{\sigma}^T F_4^T r + \bar{\rho}^T F_5^T r + \bar{\rho}^T F_6^T q) \mathrm{d}t \tag{46}$$

$$\mu = \int_0^T (r^T F_1 r + q^T F_2 q + p^T F_3 p + r^T F_4 q + r^T F_5 p + q^T F_6 p) \mathrm{d}t \tag{47}$$

where $F_1,\dots,F_6$ are matrices that depend on parameters of the system and the performance index, i.e.,

$$F_1 = M^T B_{\text{inv}}^T R B_{\text{inv}} M \tag{48}$$

$$F_2 = C^T B_{\text{inv}}^T R B_{\text{inv}} C + Q_b \tag{49}$$

$$F_3 = K^T B_{\text{inv}}^T R B_{\text{inv}} K + Q_a \tag{50}$$

$$F_4 = 2M^T B_{\text{inv}}^T R B_{\text{inv}} C \tag{51}$$

$$F_5 = 2M^T B_{\text{inv}}^T R B_{\text{inv}} K \tag{52}$$

$$F_6 = 2C^T B_{\text{inv}}^T R B_{\text{inv}} K + Q_c \tag{53}$$

where, for notational convenience, $B_{\text{inv}} = B^{-1}$. The performance index $L = L_1 + L_2$ can now be expressed as a quadratic function in terms of y:

$$L = y^T(\Delta + \Theta^T H \Theta)y + y^T\Gamma + \mu \tag{54}$$

The necessary condition of minimum L is thus

$$\frac{\mathrm{d}L}{\mathrm{d}y} = 0 \tag{55}$$

which is equivalent to

$$(\Delta + \Delta^T + 2\Theta^T H \Theta)y = -\Gamma \tag{56}$$

Equation (56) represents a system of linear algebraic equations of dimension mN with the number of equations equal to the number of unknown variables, i.e., the elements of y. It can be solved using a linear equation solver, such as a Gaussian elimination routine. In solving this equation for y, the integrals of Eqs. (45) and (46) must be evaluated. Note that the integral of Eq. (47) must be evaluated only when it is necessary to compute the value of the performance index.

The integrals of Eqs. (45–47) can be computed efficiently by writing these equations as

$$\Delta = \int_0^T (\gamma\gamma^T \otimes F_1 + \sigma\sigma^T \otimes F_2 + \rho\rho^T \otimes F_3 + \gamma\sigma^T \otimes F_4 + \gamma\rho^T \otimes F_5 + \sigma\rho^T \otimes F_6) \mathrm{d}t \tag{57}$$

$$\Gamma = \sum_{n=1}^{N} \int_0^T (2r_n \gamma \otimes f_n^1 + 2q_n \sigma \otimes f_n^2 + 2p_n \rho \otimes f_n^3 + q_n \gamma \otimes f_n^4 \; p_n \gamma \otimes f_n^5 + p_n \sigma \otimes f_n^6 + r_n \sigma \otimes g_n^4 + r_n \rho \otimes g_n^5 + q_n \rho \otimes g_n^6) \mathrm{d}t \tag{58}$$

$$\mu = \sum_{i=1}^{N} \sum_{j=1}^{N} \int_0^T (f_{ij}^1 r_i r_j + f_{ij}^2 q_i q_j + f_{ij}^3 p_i p_j + f_{ij}^4 r_i q_j + f_{ij}^5 r_i p_j + f_{ij}^6 q_i p_j) \mathrm{d}t \tag{59}$$

In Eqs. (57) and (58), the symbol $\otimes$ is a Kronecker product sign defined as

$$A \otimes F = \begin{bmatrix} f_{11}A & f_{12}A & \dots & f_{1l}A \\ f_{21}A & f_{22}A & \dots & f_{2l}A \\ \dots & \dots & \dots & \dots \\ f_{n1}A & f_{n2}A & \dots & f_{nl}A \end{bmatrix} \tag{60}$$

where A and F are matrices, with F assumed to be of dimension $n \times l$. In Eq. (58), f_i^k represents the ith column vector of F_k, and g_i^k represents the ith column vector of F_k^T. In Eq. (59), f_{ij}^k represents the ith row jth column element of F_k. Equivalently,

$$F_k = [f_1^k \quad f_2^k \quad \dots \quad f_N^k] = [f_{ij}^k] \quad \text{for} \quad k = 1,\dots,6; \quad i,j = 1,\dots,N \tag{61}$$

$$F_k^T = [g_1^k \quad g_2^k \quad \dots \quad g_N^k] \quad \text{for} \quad k = 4,5,6 \tag{62}$$

Also,

$$\sigma = \dot{\rho}, \qquad q_i = \dot{p}_i \tag{63}$$

$$\gamma = \ddot{\rho}, \qquad r_i = \ddot{p}_i \tag{64}$$

For time-invariant LQ problems, the terms associated with $F_i (i = 1,\ldots,6)$ are constants and can be taken out of the integrals. For example, the first term of Eqs. (57), (58), and (59), respectively, can be rewritten as

$$\int_0^T (\gamma^T\gamma \otimes F_1)\, dt = \left[\int_0^T (\gamma^T\gamma)\, dt\right] \otimes F_1 \tag{65}$$

$$\sum_{n=1}^{N} \int_0^T (2r_n\gamma \otimes f_n^1)\, dt = \sum_{n=1}^{N} \left[\int_0^T (2r_n\gamma)\, dt\right] \otimes f_n^1 \tag{66}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^T (f_{ij}^1 r_i r_j)\, dt = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{ij}^1 \left[\int_0^T (r_i r_j)\, dt\right] \tag{67}$$

Each of the terms in Eqs. (57–59) can be written similarly. As a consequence, only the integrals of the squares and cross products of the time functions used by the Fourier-based approach (e.g., the terms inside the brackets) need to be evaluated. These integrals are independent of the system and performance index parameters and can be solved analytically; a sample integral table is provided in Appendix B. The integral tables, which can be applied to problems of any order, make the proposed approach numerically integration-free in solving time-invariant LQ problems.

An important feature of Eq. (56) is that the coefficient matrix of y is independent of the initial conditions. Thus, for the same optimal control problem with different initial conditions, the coefficient matrix remains the same; only the right-hand side vector needs to be recomputed. As a result, numerical algorithms such as LU decomposition (and linear algebraic equation solvers based on matrix inversion) are particularly efficient for recalculation of y for different initial conditions.

Linear Quadratic Problems with Linear Boundary Constraints

This subsection applies the Fourier-based approach to solve the LQ problem for an actively controlled structural system with linear boundary constraints. In general, these constraints can be represented by

$$Ey_b = v_f \tag{68}$$

where the vector v_f and matrix E are assumed known and y_b is a subset of the parameter vector y. For example, for a problem with terminal constraints, the vector y_b can be chosen as

$$y_b = z(T) \tag{69}$$

by carrying out straightforward row and column exchanges, the performance index L of Eq. (54) can be rewritten as

$$\begin{aligned} L &= [y_a^T \quad y_b^T] \begin{bmatrix} \Delta_{aa} & \Delta_{ab} \\ \Delta_{ba} & \Delta_{bb} \end{bmatrix} \begin{bmatrix} y_a \\ y_b \end{bmatrix} + [y_a^T \quad y_b^T] \begin{bmatrix} \Gamma_a \\ \Gamma_b \end{bmatrix} + \mu \\ &= y_a^T \Delta_{aa} y_a + y_b^T \Delta_{bb} y_b + y_a^T \Delta_{ab} y_b + y_b^T \Delta_{ba} y_a + y_a^T \Gamma_a \\ &\quad + y_b^T \Gamma_b + \mu \end{aligned} \tag{70}$$

where y_a represents the vector obtained by excluding y_b from y.

The optimization problem is to minimize the quadratic function of Eq. (70) while satisfying the equality constraints of Eq. (68). Here, the solution is obtained by using a Lagrange multiplier technique. A modified performance index is written as

$$\hat{L} = L + \lambda^T(Ey_b - v_f) \tag{71}$$

where λ is a vector of Lagrange multipliers. The necessary conditions for optimality can be obtained from

$$\frac{\partial \hat{L}}{\partial y_a} = 0 \tag{72}$$

$$\frac{\partial \hat{L}}{\partial y_b} = 0 \tag{73}$$

$$\frac{\partial \hat{L}}{\partial \lambda} = 0 \tag{74}$$

Equations (72–73) are equivalent to

$$\begin{bmatrix} \Lambda_{aa} & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_{bb} \end{bmatrix} \begin{bmatrix} y_a \\ y_b \end{bmatrix} = \begin{bmatrix} -\Gamma_a \\ -\Gamma_b - E^T\lambda \end{bmatrix} \tag{75}$$

where

$$\Lambda_{aa} = \Delta_{aa} + \Delta_{aa}^T \tag{76}$$

$$\Lambda_{ab} = \Delta_{ab} + \Delta_{ba}^T \tag{77}$$

$$\Lambda_{ba} = \Delta_{ba} + \Delta_{ab}^T \tag{78}$$

$$\Lambda_{bb} = \Delta_{bb} + \Delta_{bb}^T \tag{79}$$

Using the matrix inverse lemma (see, e.g., Brogan,[7] p. 78), y_b can be expressed as

$$y_b = \Lambda_c \Gamma_a + \Lambda_d(\Gamma_b + E^T\lambda) \tag{80}$$

where

$$\Lambda_c = -\Lambda_{bb}^{-1} \Lambda_{ba}(\Lambda_{aa} - \Lambda_{ab}\Lambda_{bb}^{-1}\Lambda_{ba})^{-1} \tag{81}$$

$$\Lambda_d = -(\Lambda_{bb} - \Lambda_{ba}\Lambda_{aa}^{-1}\Lambda_{ab})^{-1} \tag{82}$$

By substituting Eq. (80) into Eq. (68) and rearranging, the Lagrange multiplier vector can be determined as

$$\lambda = (E\Lambda_d E^T)^{-1}(v_f - E\Lambda_c\Gamma_a - E\Lambda_d\Gamma_b) \tag{83}$$

With λ known, y_a and y_b can be calculated from Eq. (75).

Two special cases of linear boundary constraints are addressed next. The first case considers problems with fixed terminal conditions. The second case addresses problems with fixed initial control variables.

Case I: Problems with Fixed Terminal Conditions

Often, structural systems are required to move to target positions with prespecified velocities. For such applications, y_b is known a priori. Consequently, the necessary condition of optimality can be obtained by setting the derivatives of the performance index of Eq. (70) with respect to y_a to zero. This gives the necessary condition as

$$(\Delta_{aa} + \Delta_{aa}^T)\, y_a = -y_b^T(\Delta_{ab} + \Delta_{ba}^T) - \Gamma_a \tag{84}$$

from which the optimal value of y_a can be determined. Compared to the necessary condition of optimality of the unconstrained case represented by Eq. (56), Eq. (84) represents a system of linear algebraic equations of reduced dimension.

Case II: Problems with Fixed Initial Control Variables

Typically, optimal control problems are formulated with given initial conditions on the state variables and with the initial control variables free. In practice, actuators of physical systems are maintained at equilibrium positions when not used. As a result, initial tracking errors may occur due to the delay of the actuators to reach the optimal conditions at the beginning of the optimal trajectory. This difficulty can be overcome by using the proposed Fourier-based approach.

It is assumed that the initial control vector is specified as

$$u(0) = u_0 \tag{85}$$

From Eq. (1), the initial value of the acceleration vector can be computed from

$$\ddot{x}(0) = M^{-1}(Bu_0 - C\dot{x}_0 - Kx_0) \tag{86}$$

Like y_b in case I, $\ddot{x}(0)$ is a subset of y. As a result, the problem can be solved using the approach of case I. In summary, the proposed approach ensures that the optimal control will match the initial operating condition of the actuators.

Simulation Studies

Simulation studies were conducted to evaluate the effectiveness of the Fourier-based approach relative to standard optimal control solvers. In these studies, the configuration and control variables were generated at prespecified equally spaced points in time. Examples 1 and 2 examine time-invariant LQ problems, whereas examples 3 and 4 consider time-varying LQ problems. An empirical technique is developed and tested in example 5 to apply the Fourier-based approach to a structural system that is not actively controlled.

For the time-invariant LQ problems, the open-loop transition-matrix approach is used to first convert the TPBVP into an initial boundary-value problem via the evaluation of the exponential of the Hamiltonian matrix. The approach is adequate for implementing the optimal LQ control law if the converted initial value problem can be integrated on-line. However, off-line inspection of the system response is often required prior to any physical implementation to insure that system specifications are satisfied. As a result, the integration of the converted initial value problem is considered an essential part of LQ controller design.

The converted initial boundary-value problem is represented by a system of linear differential equations with constant coefficients. By evaluating the state transition matrix, such differential equations can be approximated by a set of difference equations from which the response of the system can be computed at equally spaced time intervals. This discrete time approach is usually more efficient than numerically integrating the differential equations. A more detailed description of this discrete time approximation can be found in Brogan[7] (pp. 272–273). In the transition-matrix approach used in examples 1 and 2, the matrix exponentials are computed numerically using the algorithm presented in Franklin and Powell.[8]

In examples 3 and 4, the open-loop approach of examples 1 and 2 is replaced by a Riccati equation solver that is computationally more efficient in solving time-varying LQ problems. The symmetry of the Riccati equation is exploited to reduce the computational cost. After the Riccati equation is integrated, the state equations are integrated to generate the response of the state and control variables. A fourth-order Runge-Kutta method (with a time step of 0.01 time unit) is used to integrate both the Riccati and state equations.

For the Fourier-based approach, a two-term Fourier-type series is assumed. The optimal parameter vector y is first obtained by solving a system of linear algebraic equations, i.e., Eq. (56), using a Gaussian elimination routine. The configuration vector and its rates are then computed from Eqs. (30), (35), and (36). The results are substituted into Eq. (39) to compute the control vector. For the time-invariant problems of examples 1 and 2, analytical integration results (summarized in integral tables, such as the sample table listed in Appendix B) are used to minimize the computational requirements. For the time-varying problems of examples 3 and 4, the integrals of Eqs. (45–47) are computed using Simpson's rule (with a time step of 0.01 time unit). To verify the accuracy of the approach, the value of the performance index from the Fourier-based approach is compared to the value from the transition-matrix or Riccati approaches. The time (in seconds) required to execute each simulation is used as an index of computational efficiency.

The computer codes used in the simulation studies were written in the C language and compiled by a Microsoft Quick C compiler (version 1.0). Efforts were made to optimize the execution speeds of these codes. The studies were conducted on a 16-MHz NEC 386 Powermate personal computer with a 16-MHz coprocessor.

Example 1

This example considers a time-invariant LQ problem for an N-degree-of-freedom structure. The mass matrix M and the control influence matrix B are $N \times N$ identify matrices $I_{N\times N}$. The damping matrix C is a band matrix of bandwidth 3 with the values of all nonzero elements equal to 1. The stiffness matrix K is also a band matrix of bandwidth 3 with diagonal elements equal to 1 and the remaining nonzero elements equal to -1. For example, for the case $N=4$, the equation of motion is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ddot{x} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \dot{x}$$

$$+ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} u \tag{87}$$

Table 1 Summary of simulation results of example 1

	Transition-matrix approach		Fourier-based approach[a]		Comparison	
N	Performance index	Time	Performance index	Time	%Time[b]	$\Delta\% PI$[c]
2	62.841290	1.10	62.841291	0.77	70.0	$<1.0\times10^{-6}$
3	196.687742	2.96	196.687743	1.21	40.9	$<1.0\times10^{-6}$
4	427.405946	6.26	427.405950	1.84	29.4	$<1.2\times10^{-6}$
5	806.240859	11.59	806.240870	2.75	23.7	$<1.3\times10^{-6}$
6	1360.441759	19.17	1360.441781	4.01	20.9	$<1.6\times10^{-6}$
7	2123.257824	29.55	2123.257836	5.71	19.3	$<1.9\times10^{-6}$
8	3127.938234	43.29	3127.938295	7.80	18.0	$<2.0\times10^{-6}$
9	4407.732165	60.69	4407.732257	10.44	17.2	$<2.1\times10^{-6}$
10	5995.888795	82.05	5995.888925	13.62	16.6	$<2.2\times10^{-6}$

[a]With two-term Fourier-type series.
[b]Percent of execution time of Fourier-based approach relative to execution time of transition-matrix approach.
[c]Percent difference of performance index of Fourier-based approach relative to performance index of transition-matrix approach.

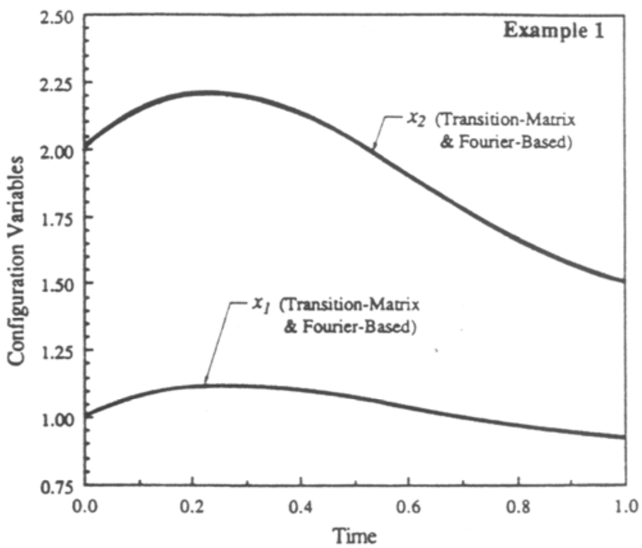

Fig. 1a History of configuration variables for example 1.

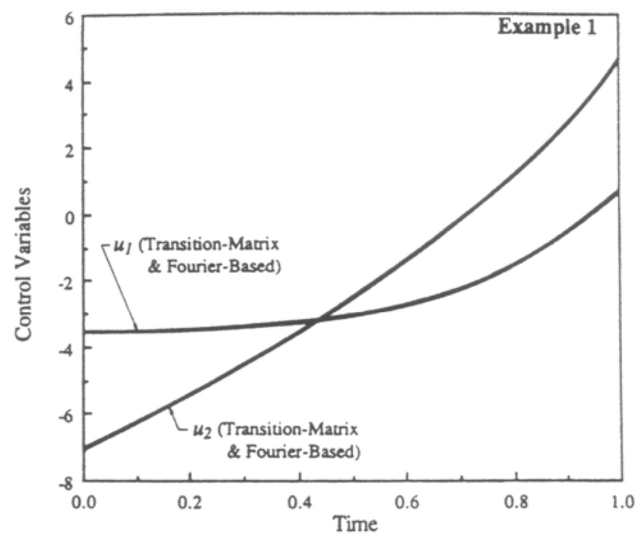

Fig. 1b History of control variables for example 1.

Table 2 Summary of simulation results of example 2

	Transition-matrix approach		Fourier-based approach[a]		Comparison	
N	Performance index	Time	Performance index	Time	%Time[b]	$\Delta\% PI$[c]
2	164.059510	1.21	164.059510	0.72	59.5	$<3.6\times10^{-7}$
3	517.597140	3.13	517.597141	1.10	31.9	$<3.9\times10^{-7}$
4	1183.103282	6.65	1183.103286	1.65	24.8	$<4.2\times10^{-7}$
5	2258.102938	12.20	2258.102948	2.37	19.4	$<4.9\times10^{-7}$
6	3840.121538	20.05	3840.121558	3.31	16.5	$<5.5\times10^{-7}$
7	6026.684485	30.70	6026.684521	4.56	14.9	$<6.2\times10^{-7}$
8	8915.317185	44.71	8915.317243	6.10	18.0	$<6.7\times10^{-7}$
9	12603.545041	62.40	12603.545129	7.97	13.6	$<7.1\times10^{-7}$
10	17188.893458	84.09	17188.893583	10.27	12.2	$<7.4\times10^{-7}$

[a]With two-term Fourier-type series.
[b]Percent of execution time of Fourier-based approach relative to execution time of transition-matrix approach.
[c]Percent difference of performance index of Fourier-based approach relative to performance index of transition-matrix approach.

The initial conditions of the N-degree-of-freedom structure are taken as

$$x^T(0)=\dot{x}^T(0)=[1 \quad 2 \quad 3 \quad \dots \quad N] \tag{88}$$

The weighting matrices for the performance index, Eq. (2), are defined as

$$H=10I_{N\times N} \tag{89}$$

$$R=I_{N\times N} \tag{90}$$

$$Q=I_{N\times N} \tag{91}$$

The final time T is taken as 1 time unit.

The simulation results for $N=2,\dots,10$ are summarized in Table 1. (Note that the unit for execution time is seconds.) In every case, the value of the performance index of the Fourier-based approach has a percentage error less than 10^{-5} relative to the performance index value obtained by the transition-matrix approach. As shown in Table 1, the Fourier-based approach is more efficient for high-order systems. This is because the computational requirements for setting up the analytical integration results (e.g., the integral table listed in Appendix B) are fixed and become less significant for higher-order systems.

The time histories of x and u for the case $N=2$ are plotted in Figs. 1a and 1b, respectively. The figures show that the solutions of the two different approaches are indistinguishable. This agreement was observed in all other cases studied in this example.

Example 2

This example investigates the effectiveness of the Fourier-based approach in handling LQ problems with fixed terminal states. The problem specifications of this example are identical to those of example 1 with two exceptions. The first difference is that the terminal weighting matrix H is specified as a null matrix. The second difference is that the structure is required to reach the origin with zero velocity at the terminal time, i.e.,

$$x(1)=\dot{x}(1)=0 \tag{92}$$

The simulations results for $N=2,\dots,10$ are summarized in Table 2. The table shows that the Fourier-based approach is more efficient than the transition-matrix approach, especially in handling high-order systems. The values of the performance index indicate that the percentage error of the Fourier-based performance index is less than 10^{-5} in all cases. In comparison with Table 1, Table 2 indicates that slightly higher efficiency is achieved in example 2. As suggested by Eq. (84), there are fewer unknown configuration variable parameters than in example 1. The time histories of x and u for the case $N=2$ are plotted in Figs. 2a and 2b, respectively. Again, the solutions of the two approaches match closely.

Example 3

The problem specifications of this example are identical to those of example 1 except that the diagonal terms of the mass matrix are changed to $1-0.1t$. As a consequence, the problem becomes time-varying.

This example was solved via the Riccati and Fourier-based approaches. The simulation results for $N=2,\ldots,10$ are summarized in Table 3. As shown in the table, the Fourier-based approach requires less computation time than the Riccati equation solver in all cases except for the case $N=2$. The constant cost of computing the time functions used by the Fourier-based approach becomes less significant in generatingthe optimal trajectories for high-order systems. As such, the Fourier-based approach is more efficient in handling high-order systems.

The time histories of x and u for the case N-2 are plotted in Figs. 3a and 3b, respectively. As with the previous figures, the solutions of the two different approaches are indistinguishable.

Example 4

The goal of this example is to investigate the robustness of the Fourier-based approach when subject to heavy terminal weighting. A heavy penalty is often placed on the terminal states when an unconstrained LQ solver for problems with-fixed terminal states is used. However, as demonstrated in the following paragraphs, this technique may cause numerical instability when a Riccati-based approach is used.

First, consider the case $N=2$ in example 3 except that the H matrix is modified to have diagonal elements of 200 (compared to 10, previously). The performance index value obtained by the Fourier-based approach is 151.445506. Performance index values computed by a Riccati equation solver using various of integration time steps are summarized in Table 4. As shown in the table, the Riccati equation solver suffers from a problem of numerical instability when the integration time step is not sufficiently small. Although this difficulty can be alleviated by reducing the integration step size, this increases the computational cost. The problem of numerical instability becomes worse for the Riccati-based approach as the terminal weighting increases.

To further test the robustness of the Fourier-based approach, the case $N=2$ of the fixed terminal condition problem of example 2 is reconsidered. Here, the problem is treated as an unconstrained LQ problem with free boundary conditions and with a nonzero diagonal terminal weighting matrix H. Simulation results for various H matrices are summarized in Table 5. The table shows that the value of the integral part of the performance index converges to the performance index value obtained in example 2 as the magnitude of the terminal weighting increases. These results indicate that the accuracy of the Fourier-based approach is insensitive to large terminal weightings.

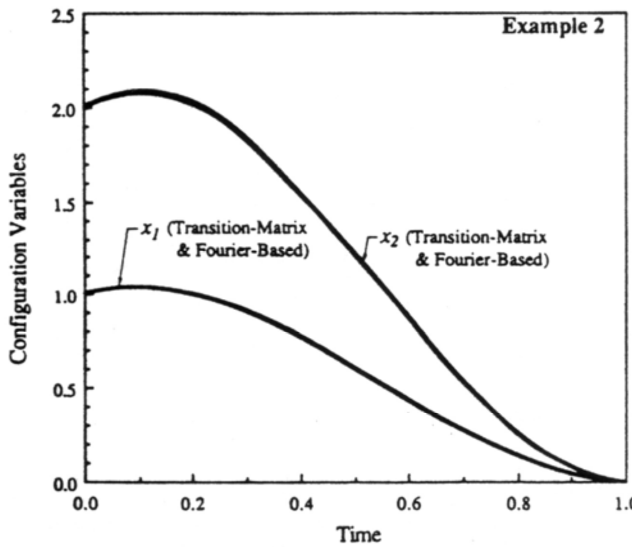

Fig. 2a History of configuration variables for example 2.

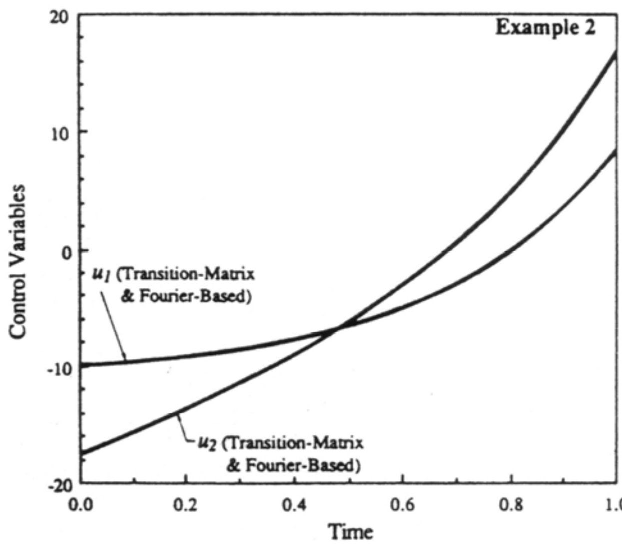

Fig. 2b History of control variables for example 2.

Table 3 Summary of the simulation results of example 3

	Riccati-based approach		Fourier-based approach[a]		Comparison	
N	Performance index	Time	Performance index	Time	%Time[b]	Δ%PI[c]
2	63.014689	9.61	63.014691	11.15	116.0	$<3.1\times10^{-6}$
3	191.391620	29.61	191.391626	23.61	79.7	$<3.2\times10^{-6}$
4	429.125200	74.13	429.125213	40.92	55.2	$<3.3\times10^{-6}$
5	809.605772	155.94	809.605798	65.30	42.9	$<3.4\times10^{-6}$
6	1366.230023	294.62	1366.230068	95.30	32.4	$<3.4\times10^{-6}$
7	2132.394458	509.33	2132.394530	130.44	25.6	$<3.5\times10^{-6}$
8	3141.495577	828.71	3141.495684	172.52	20.8	$<3.5\times10^{-6}$
9	4426.929882	1272.02	4426.930036	225.69	17.7	$<3.5\times10^{-6}$
10	6022.093875	1885.87	6022.094086	279.14	14.8	$<3.6\times10^{-6}$

[a]With two-term Fourier-type series.
[b]Percent of execution time of Fourier-based relative to execution time of Riccati-based approach.
[c]Percent difference of performance index of Fourier-based approach relative to performance index of Riccati-based approach.

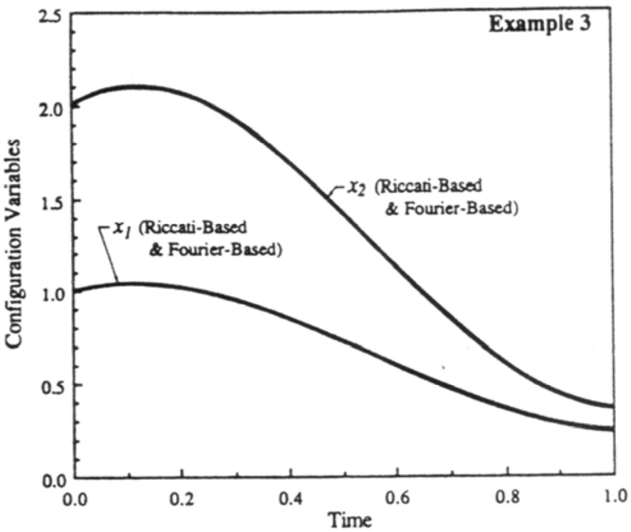

Fig. 3a History of configuration variables for example 3.

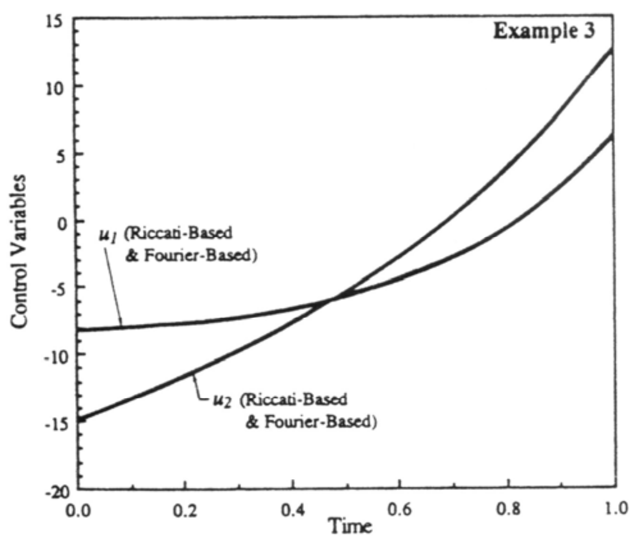

Fig. 3b History of control variables for example 3.

It should be noted that the open-loop transition-matrix approach is also very robust under heavy terminal weighting, although evaluation of the transition matrix for the time-varying Hamiltonian matrix is a computationally intensive process. In contrast, the Fourier-based approach is efficient and robust in handling heavy terminal weighting for both time-varying and time-invariant LQ problems.

Example 5

The goal of this example is to illustrate an empirical technique that generalizes the Fourier-based approach to structural systems that are not actively controlled, i.e., systems with fewer control variables than degrees of freedom. Here, a single-input, two-degree-of-freedom structural system is considered. The equation of motion is represented by

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \ddot{x} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \dot{x} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{93}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{94}$$

Table 4 Summary of Riccati-based simulation results of example 4

Time step	Performance index[a]
1/100	Unstable
1/200	151.358291
1/300	151.442453
1/400	151.445638
1/500	151.445728
1/600	151.445649
1/700	151.445592
1/800	151.445558
1/900	151.445538
1/1000	151.445525
1/2000	151.445501

[a]Compares to performance index of 151.445506 of Fourier-based approach.

Table 5 Summary of Fourier-based simulation results of example 4

H_{ii}[a]	Performance Index (L)	Integral part of PI (L_2)
10^1	64.841290	28.996032
10^2	138.521335	117.355700
10^3	161.039320	158.080595
10^4	163.751807	163.444801
10^5	164.028629	163.997814
10^6	164.056369	164.053286
10^7	164.059143	164.058835
10^8	164.059420	164.059390
10^9	164.059448	164.059445
10^{10}	164.059510	164.059510
10^{20}	164.059510	164.059510
10^{30}	164.059510	164.059510

[a]Diagonal elements of terminal weighting matrix.

The performance index is

$$L = 10x_1^2(1) + 10x_2^2(1) + \int_0^1 (x^T Q_1 x + \dot{x}^T Q_2 \dot{x} + \dot{x}^T Q_3 x + u^2) \mathrm{d}t \tag{95}$$

with

$$Q_1 = Q_2 = Q_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{96}$$

To solve this unconstrained LQ problem using the Fourier-based approach, an artificial control variable u^* is introduced as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \ddot{x} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \dot{x} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^* \\ u \end{bmatrix} \tag{97}$$

The performance index is also modified as

$$L^* = 10x_1^2(1) + 10x_2^2(1) + \int_0^1 (x^T Q_1 x + \dot{x}^T Q_2 \dot{x} + \dot{x}^T Q_3 x + u^2 + R u^{*2}) \, \mathrm{d}t \tag{98}$$

A large magnitude of the weighting coefficient R will make u^* small; hence, the solution of the modified optimal control problem will closely approximate the solution of the original optimal control problem. Here, a value of $R = 10{,}000$ was selected (arbitrarily) and a two-term Fourier-type series was used. The results of the configuration and control variable time histories are shown in Figs. 4a and 4b, respectively. The results from the Fourier-based and transition-matrix approaches agree very well.

Although it may seem that the accuracy of the solution may be improved simply by employing larger R in the modified performance index, this may cause numerical problems since the control weighting matrix may become ill-conditioned. To overcome such problems, Yen and Nagurka[9] developed an optimization technique to simultaneously minimize the performance index and the contribution of the artificial control variables.

In most cases, the preceding empirical technique is capable of providing satisfactory predictions of the optimal solution and is thus adequate for use as a software design tool. In summary, using large but not extremely large artificial control weightings can satisfy the need for off-line LQ controller design without causing numerical problems.

Discussion

The Fourier-based approach is a state parameterization approach. As shown in Eq. (27), the trajectory of each configuration variable is a function of the boundary values of the configuration variable and its first and second derivatives as well as the coefficients of the Fourier-type series. An advantage of state parameterization is that it characterizes the optimal trajectory (which, in theory, consists of an infinite number of points) by a relatively small number of state parameters. An optimal control problem can thus be converted into an algebraic optimization problem (i.e., a mathematical programming problem). The optimal values of the state parameters can be determined to minimize the value of the performance index. In general, the corresponding computations are much less complicated than those involved in standard optimal control solvers.

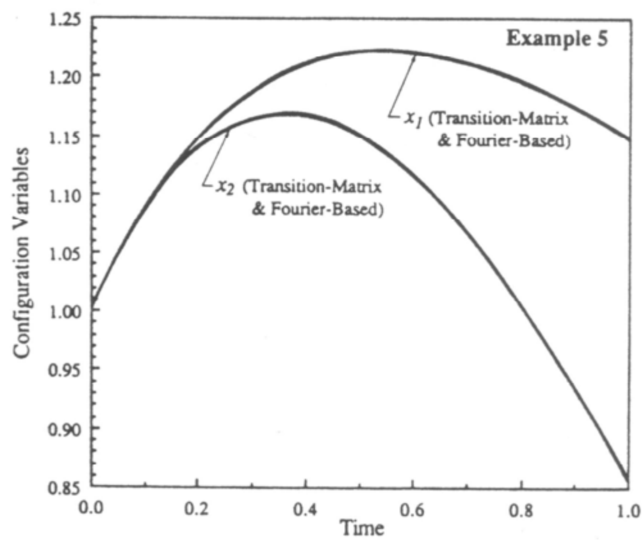

Fig. 4a History of configuration variables for example 5.

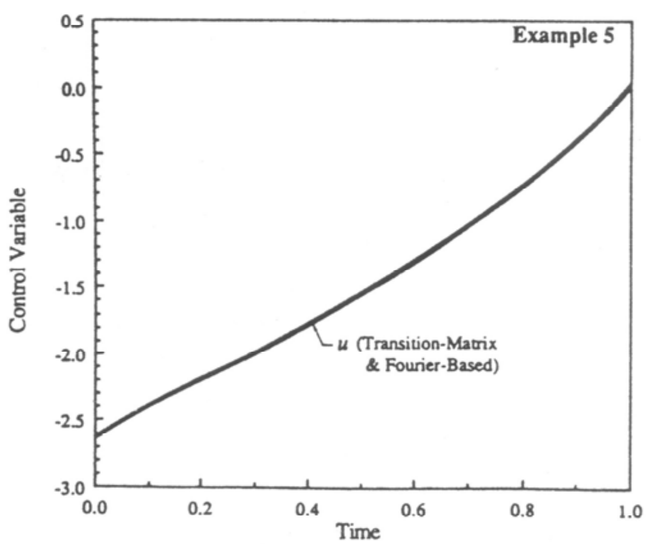

Fig. 4b History of control variable for example 5.

For LQ control of structural systems, the performance index, which initially is written as a quadratic functional (i.e., a function of configuration and control variables, which themselves are functions of time), is converted into a quadratic function (i.e., a function of time-independent parameters of the Fourier-based approach). By differentiating this quadratic function with respect to the free parameters, the necessary condition of optimality is derived as a system of linear algebraic equations that can be solved readily. As verified by simulation results, this Fourier-based approach is in most cases computationally more efficient than standard LQ problem solvers in handling time-invariant and time-varying LQ problems. The computational advantage of the Fourier-based approach is especially evident for high-order systems.

In implementing the Fourier-based method, finite-term Fourier-type series are employed. Consequently, the Fourier-based approach can be classified as a near-optimal state parameterization approach. In each of the simulation studies, only a two-term Fourier-type series was used. The success of a two-term series suggests that the higher-order terms of the Fourier-type series do not contribute significantly to the quality of the near-optimal solution. As a consequence, in many cases a Fourier-type series of few terms is required to achieve satisfactory results.

The accuracy of the Fourier-based approach can be estimated empirically by increasing the number of terms of the Fourier-type series [i.e., incrementing K of Eq. (16)] and comparing the value of the performance index. Additional terms can be added, on a term-by-term basis, until the value of the performance index converges, indicating that the optimal solution has been reached.

Work is currently underway to apply the Fourier-based approach to solve LQ problems with constraints on configuration and/or control variables. By converting these constraints into systems of algebraic inequalities, the optimal values of the configuration parameters of the Fourier-based approach can be determined by nonlinear programming.[5] If the configuration and control constraints are linear, the resulting algebraic inequalities will also be linear. As a result, a constrained LQ problem can be converted into a quadratic programming problem that can be solved by a number of well developed algorithms.

Work is also in progress to generalize the proposed Fourier-based approach for solving nonlinear optimal control problems. The underlying idea is to convert a nonlinear optimal control problem into a sequence of LQ problems via a quasilinearization approach.[10] Each of the LQ problems is then solved by the Fourier-based approach of this paper. This procedure was demonstrated for solving nonlinear optimal control problems of one- and two-degree-of-freedom robotic manipulators.[11] Simulation results show that the approach is robust and efficient in generating optimal trajectories for low-order manipulator problems with quadratic performance indices. Currently, this approach is being applied to solve higher-order manipulator problems and to solve general nonlinear optimal control problems with and without constraints.

Conclusions

This paper presents a computationally efficient alternative to standard approaches for the solution of the optimal trajectories of linear structural systems with quadratic performance indices. The approach relies on a Fourier-based approximation of the configuration vector and converts the optimal control problem into a simple mathematical programming problem.

The proposed approach avoids formulation of the costate equations and requires no numerical integration in solving time-invariant LQ problems. A major advantage of this approach is its computational efficiency especially in handling high-order problems. A second advantage of the approach is that it is applicable to problems with free, fixed, or linearly constrained boundary conditions. In particular, the boundary value of the control vector can be set to any arbitrary level to match the operating conditions of the actuators. This feature does not seem to be feasible in previous approaches.

Appendix A: Coefficients of Auxiliary Polynomial

In the Fourier-based approach, the ith generalized coordinate $x_i(t)$ is represented by the sum of a fifth-order auxiliary polynomial and a finite-term Fourier-type series as follows:

$$x_i(t) = \eta_i(t) + \pi_i(t) \tag{A1}$$

where

$$\eta_i(t) = \sum_{k=0}^{5} d_{ik} t^k \tag{A2}$$

$$\pi_i(t) = \sum_{k=1}^{K} \left(a_{ik} \cos \frac{2k\pi t}{T} + b_{ik} \sin \frac{2k\pi t}{T} \right) \tag{A3}$$

By writing boundary-value equations for the configuration variable and its rates,

$$x_{i0} \equiv x_i(0) = \eta_i(0) + \pi_i(0) \tag{A4}$$

$$\dot{x}_{i0} \equiv \dot{x}_i(0) = \dot{\eta}_i(0) + \dot{\pi}_i(0) \tag{A5}$$

$$\ddot{x}_{i0} \equiv \ddot{x}_i(0) = \ddot{\eta}_i(0) + \ddot{\pi}_i(0) \tag{A6}$$

$$x_{iT} \equiv x_i(T) = \eta_i(T) + \pi_i(T) \tag{A7}$$

$$\dot{x}_{iT} \equiv \dot{x}_i(T) = \dot{\eta}_i(T) + \dot{\pi}_i(T) \tag{A8}$$

$$\ddot{x}_{iT} \equiv \ddot{x}_i(T) = \ddot{\eta}_i(T) + \ddot{\pi}_i(T) \tag{A9}$$

the coefficients of the polynomial can be expressed as functions of the boundary values and the coefficients of the Fourier-type series. That is,

$$d_{i0} = x_{i0} - \sum_{k=1}^{K} a_{ik} \tag{A10}$$

$$d_{i1} = \dot{x}_{i0} - T^{-1} \sum_{k=1}^{K} \upsilon_k b_{ik} \tag{A11}$$

$$d_{i2} = \frac{1}{2} \left(\ddot{x}_{i0} + T^{-2} \sum_{k=1}^{K} \upsilon_k^2 a_{ik} \right) \tag{A12}$$

$$d_{i3} = \left[10(-x_{i0} + x_{iT}) - \sum_{k=1}^{K} \upsilon_k^2 a_{ik} + 10 \sum_{k=1}^{K} \upsilon_k b_{ik} \right] T^{-3} + (-6\dot{x}_{i0} - 4\dot{x}_{iT}) T^{-2} + \left(-\frac{3}{2} \ddot{x}_{i0} + \frac{1}{2} \ddot{x}_{iT} \right) T^{-1} \tag{A13}$$

$$d_{i4} = \left[15(x_{i0} - x_{iT}) + + \frac{1}{2} \sum_{k=1}^{K} \upsilon_k^2 a_{ik} - 15 \sum_{k=1}^{K} \upsilon_k b_{ik} \right] T^{-4} + (8\dot{x}_{i0} + 7\dot{x}_{iT}) T^{-3} + \left(\frac{3}{2} \ddot{x}_{i0} - \ddot{x}_{iT} \right) T^{-2} \tag{A14}$$

$$d_{i5} = \left[6(-x_{i0} + x_{iT}) + 6 \sum_{k=1}^{K} \upsilon_k b_{ik} \right] T^{-5} + 3(-\dot{x}_{i0} - \dot{x}_{iT}) T^{-4} - \frac{1}{2} (\ddot{x}_{i0} - \ddot{x}_{iT}) T^{-3} \tag{A15}$$

where

$$\upsilon_k = 2k\pi \tag{A16}$$

Appendix B: Sample Integral Table

Evaluation of the elements of $\int_0^T \rho^T \rho \, dt$

φ	$\int_0^T \varphi \, dt$
ρ_1^2	$\frac{T^5}{9240}$
$\rho_1\rho_2$	$\frac{181}{55,440} T^3$
$\rho_1\rho_3$	$\frac{-13}{13,860} T^4$
$\rho_1\rho_4$	$\frac{1}{11,088} T^3$
$\rho_1\alpha_k$	$\left(\frac{-1}{120} + \frac{\upsilon_k^2}{5040} - \frac{6}{\upsilon_k^4} \right) T^3$
$\rho_1\beta_k$	$\left(\frac{-7\upsilon_k}{55,440} - \frac{1}{\upsilon_k^3} + \frac{60}{\upsilon_k^5} \right) T^3$
ρ_2^2	$\frac{181}{462} T$
$\rho_2\rho_3$	$\frac{-311}{4620} T^2$
$\rho_2\rho_4$	$\frac{281}{55,440} T^3$
$\rho_2\alpha_k$	$\left(-\frac{1}{2} + \frac{\upsilon_k^2}{120} \right) T$
$\rho_2\beta_k$	$\left(\frac{8\upsilon_k}{231} - \frac{1}{\upsilon_k} - \frac{720}{\upsilon_k^5} \right) T^3$
ρ_3^2	$\frac{52}{3465} T$
$\rho_3\rho_4$	$\frac{-23}{18,480} T^4$
$\rho_3\alpha_k$	$\left(\frac{1}{10} - \frac{11}{5040} \upsilon_k^2 + \frac{1}{\upsilon_k^2} + \frac{12}{\upsilon_k^4} \right) T^2$
$\rho_3\beta_k$	$\left(\frac{-5}{924} \upsilon_k + \frac{360}{\upsilon_k^5} \right) T^2$
ρ_4^2	$\frac{T^5}{9240}$
$\rho_4\alpha_k$	$\left(\frac{-1}{120} + \frac{\upsilon_k^2}{5040} - \frac{6}{\upsilon_k^4} \right) T^3$
$\rho_4\beta_k$	$\left(\frac{17\upsilon_k}{55,440} + \frac{1}{\upsilon_k^3} - \frac{60}{\upsilon_k^5} \right) T^3$
α_k^2	$\left(\frac{3}{2} - \frac{\upsilon_k^2}{30} + \frac{\upsilon_k^4}{2520} + \frac{24}{\upsilon_k^2} \right) T$
$\alpha_i\alpha_k,\ i \neq k$	$\left[1 - \frac{\upsilon_i^2 + \upsilon_k^2}{60} + \frac{\upsilon_i^2\upsilon_k^2}{2520} - 12\left(\frac{\upsilon_i^2}{\upsilon_k^4} + \frac{\upsilon_k^2}{\upsilon_i^4} \right) \right] T$
β_k^2	$\left(\frac{1}{2} + \frac{5}{462} \upsilon_k^2 - \frac{1440}{\upsilon_k^4} \right) T$
$\beta_i\beta_k,\ i \neq k$	$\left[\frac{5\upsilon_i\upsilon_k}{462} - 720 \left(\frac{\upsilon_i}{\upsilon_k^5} + \frac{\upsilon_k}{\upsilon_i^5} \right) \right] T$

$\upsilon_k = 2k\pi$

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