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A FOURIER-BASED OPTIMAL CONTROL APPROACH FOR STRUCTURAL SYSTEMS

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ABSTRACT

This paper considers the optimal control of structural systems with quadratic performance indices. The proposed approach approximates each configuration variable of a structural model by the sum of a fifth order polynomial and a finite term Fourier-type series. In contrast to standard linear optimal control approaches which typically require the solution of Riccati equations, the method adopted here is a near optimal approach in which the necessary and sufficient condition of optimality is derived as a system of linear algebraic equations. These equations can be solved directly by a method such as Gaussian elimination. The proposed approach is computationally efficient and can be applied to structural systems of high dimension and/or to structural systems with fixed (or highly penalized) terminal states without numerical difficulties.

INTRODUCTION

The optimal control of structural systems has important applications including large space structures and civil engineering structures. In the literature, these systems are typically modeled as linear, second-order differential equations.

The optimal control of a linear dynamical system with a quadratic performance index is usually solved by the Hamilton-Jacobi approach. Mathematically, this approach is a variational method which usually requires the solution of a differential matrix Riccati equation with a terminal condition. Various algorithms have been proposed to solve this type of equation; an extensive reference list can be found in [1]. In the interest of achieving real-time implementation, these algorithms have usually been designed for improved computational efficiency, since computational "bottlenecks" typically arise in solving for the optimal control of high dimension systems.

Without resorting to variational methods, Yen and Nagurka [2] have proposed a Fourier-based approach to generate near optimal trajectories of general dynamical systems. The basic idea of this approach is to represent the time history of each generalized coordinate by an auxiliary polynomial and a finite-term Fourier-type series. The free variables, such as the (free) coefficients of the polynomial and the Fourier-type series, are adjusted by a nonlinear programming method such that the performance index is minimized. The effectiveness of this technique has been demonstrated by simulation studies [2,3].

This research specializes this Fourier-based approach to linear systems with quadratic performance indices. The method exploits the linearity of the system model and the quadratic nature of the performance index to guarantee identification of a global minimum. In simulation studies, the method proves accurate and demonstrates impressive computational efficiency.

METHODOLOGY

The behavior of a controlled linear structure is governed by the system of differential equations:

$$\underline{M}\ddot{\underline{x}}(t) + \underline{C}\dot{\underline{x}}(t) + \underline{K}\underline{x}(t) = \underline{B}\underline{u}(t) \qquad (1)$$

with initial conditions $\underline{x}(0) = \underline{x}_0$, $\dot{\underline{x}}(0) = \dot{\underline{x}}_0$, where $\underline{x}$ is an $N \times 1$ configuration vector (*i.e.*, a column vector of N configuration variables), $\underline{u}$ is an $L \times 1$ control vector, $\underline{M}$ is an $N \times N$ positive definite mass matrix, $\underline{C}$ is an $N \times N$ positive semidefinite structural damping matrix, $\underline{K}$ is an $N \times N$ positive semidefinite stiffness matrix, and $\underline{B}$ is an $N \times L$ control influence matrix.

In this paper, it is assumed that $L \leq N$, *i.e.*, the number of control variables is less than or equal to the number of configuration variables. The derivation below and Example 1 consider the case $L = N$, *i.e.*, the configuration and control vectors have the same dimension, and $\underline{B}$ is nonsingular. For this case, the structure is actively controlled. Example 2 addresses the case $L < N$.

The design goal is to find the optimal control $\underline{u}(t)$ in the time interval $[0, t_f]$ such that the quadratic performance index

$$J = \begin{bmatrix} \underline{x}(t_f) \\ \dot{\underline{x}}(t_f) \end{bmatrix}^T \underline{H} \begin{bmatrix} \underline{x}(t_f) \\ \dot{\underline{x}}(t_f) \end{bmatrix} + \int_0^{t_f} [\underline{x}^T \underline{Q}_1 \underline{x} + \dot{\underline{x}}^T \underline{Q}_2 \dot{\underline{x}} + \dot{\underline{x}}^T \underline{Q}_3 \underline{x} + \underline{u}^T \underline{R}\underline{u}]\, dt \qquad (2)$$

is minimized. Here, $\underline{Q}_1$, $\underline{Q}_2$, $\underline{Q}_3$, and $\underline{H}$ are real, positive, semidefinite matrices and $\underline{R}$ is a positive definite matrix. In addition, $\underline{H}$ and $\underline{R}$ are symmetric. (T denotes transpose.) It is assumed that the configuration and control vectors are not bounded, the terminal configuration $\underline{x}(t_f)$ and its rates $\dot{\underline{x}}(t_f)$ and $\ddot{\underline{x}}(t_f)$ are free, and the terminal time t_f is fixed.

The basic idea of the proposed approach is to approximate each configuration variable by the sum of an auxiliary polynomial and a finite term Fourier-type series, *i.e.*, for $i = 1, \ldots, N$,

$$x_i(t) = d_i(t) + \sum_{k=1}^{K} [a_{ik} \cos\frac{2k\pi t}{t_f} + b_{ik} \sin\frac{2k\pi t}{t_f}] \quad (3)$$

where K is the number of terms included in the Fourier-type series and d_i is a fifth-order polynomial in time

$$d_i(t) = d_{i0} + d_{i1}t + d_{i2}t^2 + d_{i3}t^3 + d_{i4}t^4 + d_{i5}t^5 \quad (4)$$

The six coefficients of this auxiliary polynomial can be written in terms of six boundary conditions, *i.e.*, initial conditions $x_i(0)$, $\dot{x}_i(0)$, and $\ddot{x}_i(0)$, and terminal conditions $x_i(t_f)$, $\dot{x}_i(t_f)$, and $\ddot{x}_i(t_f)$. Explicit expressions for these coefficients are given in [2].

Equation (3) can be rearranged and presented in the form

$$x_i(t) = p_i + \rho_1 x_{if} + \rho_2 x_{if} + \rho_3 \dot{x}_{if} + \rho_4 \ddot{x}_{if} + \sum_{k=1}^{K} \alpha_k a_{ik} + \sum_{k=1}^{K} \beta_k b_{ik} \quad (5)$$

where $x_{i0} = x_i(0)$, $x_{if} = x_i(t_f)$, and similarly for the corresponding time derivatives, and where

$$p_i = x_{i0} + x_{i0}t + [-10x_{i0} - 6\dot{x}_{i0}t_f](t/t_f)^3 + [15x_{i0} + \dot{x}_{i0}t_f](t/t_f)^4 + [-6x_{i0} - 3\dot{x}_{i0}t_f](t/t_f)^5 \quad (6)$$

$$\rho_1 = \frac{1}{2}t_f^2[(t/t_f)^2 - 3(t/t_f)^3 + 3(t/t_f)^4 - (t/t_f)^5] \quad (7)$$

$$\rho_2 = [10(t/t_f)^3 - 15(t/t_f)^4 + 6(t/t_f)^5] \quad (8)$$

$$\rho_3 = t_f[-4(t/t_f)^3 + 7(t/t_f)^4 - 3(t/t_f)^5] \quad (9)$$

$$\rho_4 = \frac{1}{2}t_f^2[(t/t_f)^3 - 2(t/t_f)^4 + (t/t_f)^5] \quad (10)$$

$$\alpha_k = -1 + 4k^2\pi^2[(t/t_f)^2 - 2(t/t_f)^3 + (t/t_f)^4] + \cos\frac{2k\pi t}{t_f} \quad (11)$$

$$\beta_k = 2k\pi[-(t/t_f) + 10(t/t_f)^3 - 15(t/t_f)^4 + 6(t/t_f)^5] + \sin\frac{2k\pi t}{t_f} \quad (12)$$

Since the initial conditions x_{i0} and $\dot{x}_{i0}$ are presumed given, p_i is a known function of time. Furthermore, the parameters defined in equations (7)-(12) are configuration independent and are functions of time only, since the terminal time t_f is assumed known.

From equation (5), the configuration variable $x_i(t)$ can be written in compact form as

$$x_i(t) = p_i + \underline{\rho}^T \underline{y}_i \quad (13)$$

where

$$\underline{\rho}^T = [\rho_1 \ \rho_2 \ \rho_3 \ \rho_4 \ \alpha_1 \ldots \alpha_K \ \beta_1 \ldots \beta_K] \quad (14)$$

and

$$\underline{y}_i = [x_{i0} \ x_{if} \ \dot{x}_{if} \ \ddot{x}_{if} \ a_{i1} \ldots a_{iK} \ b_{i1} \ldots b_{iK}]^T \quad (15)$$

are vectors of dimension $m = 4 + 2K$.

The configuration variables for the N degrees of freedom can be written in terms of a configuration vector $\underline{x}(t)$, *i.e.*,

$$\underline{x}(t) = \underline{p}(t) + \underline{\rho}^*(t)\underline{y} \quad (16)$$

where

$$\underline{x}(t) = [x_1(t) \quad x_2(t) \ \ldots \ x_N(t)]^T \quad (17)$$

$$\underline{p}(t) = [p_1(t) \quad p_2(t) \ \ldots \ p_N(t)]^T \quad (18)$$

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_N \end{bmatrix} \quad \underline{\rho}^* = \begin{bmatrix} \underline{\rho}^T & \underline{0} & \ldots & \underline{0} \\ \underline{0} & \underline{\rho}^T & \ldots & \underline{0} \\ & & & \\ \underline{0} & \ldots & \ldots & \underline{\rho}^T \end{bmatrix} \quad (19),(20)$$

Note that $\underline{y}$ is a column vector of dimension Nm and that $\underline{\rho}^*$ is a matrix of dimension $N \times Nm$. Similarly, configuration rate vectors,

$$\dot{\underline{x}}(t) = \underline{q}(t) + \underline{\sigma}^*(t)\underline{y} \quad (21)$$

and

$$\ddot{\underline{x}}(t) = \underline{r}(t) + \underline{\phi}^*(t)\underline{y}, \quad (22)$$

can be introduced, where

$$\underline{q} = \dot{\underline{p}} \quad , \quad \underline{\sigma}^* = \dot{\underline{\rho}}^* \quad (23)$$

$$\underline{r} = \ddot{\underline{p}} \quad , \quad \underline{\phi}^* = \ddot{\underline{\rho}}^* \quad (24)$$

Since $\underline{x}(t)$, $\dot{\underline{x}}(t)$, and $\ddot{\underline{x}}(t)$ are known functions of $\underline{y}$, the control vector $\underline{u}(t)$ can be expressed from equation (1) as a function of $\underline{y}$. Ultimately, the interest is to express the performance index as a function of $\underline{y}$. Toward this end, the performance index of equation (2) is decomposed into two parts:

$$J = J_1 + J_2 \quad (25)$$

where J_1 is the cost associated with the terminal configuration and its rate and J_2 is the cost associated with the trajectory. The terminal configuration and its rate can be written as a linear transformation of $\underline{y}$, *i.e.*,

$$\begin{bmatrix} \underline{x}(t_f) \\ \underline{\dot{x}}(t_f) \end{bmatrix} = \underline{Z}\underline{y} \quad (26)$$

where $\underline{Z}$ is a $2N \times mN$ matrix with elements 1 and 0, specified according to

$$z_{ij} = \begin{cases} 1 \ , & j=(i-1)m+2 \quad for \ i=1,...,N \\ & j=(i-N-1)m+3 \quad for \ i=N+1,...,2N \\ 0 \ , & otherwise \end{cases} \quad (27)$$

From equation (26), the cost J_1 is

$$J_1 = \begin{bmatrix} \underline{x}(t_f) \\ \underline{\dot{x}}(t_f) \end{bmatrix}^T \underline{H} \begin{bmatrix} \underline{x}(t_f) \\ \underline{\dot{x}}(t_f) \end{bmatrix} = \underline{y}^T\underline{Z}^T\underline{H}\underline{Z}\underline{y} \quad (28)$$

From equation (1) (for the control vector) and equations (16), (21), and (22) (for the configuration vector and its rates), the cost J_2 is

$$J_2 = \int_0^{t_f}[\underline{x}^T\underline{Q}_1\underline{x}+\underline{\dot{x}}^T\underline{Q}_2\underline{\dot{x}}+\underline{\dot{x}}^T\underline{Q}_3\underline{x}+\underline{u}^T\underline{R}\underline{u}]dt \quad (29)$$

$$= \int_0^{t_f}[\underline{y}^T\underline{\Lambda}\underline{y} + \underline{y}^T\underline{\Gamma} + \underline{\Omega}^T\underline{y} + \underline{\Sigma}]dt \quad (30)$$

In equation (30)

$$\underline{\Lambda} = \underline{\phi}^{*T}\underline{F}_1\underline{\phi}^* + \underline{\sigma}^{*T}\underline{F}_2\underline{\sigma}^* + \underline{\rho}^{*T}\underline{F}_3\underline{\rho}^* + \underline{\phi}^{*T}\underline{F}_4\underline{\sigma}^* + \underline{\phi}^{*T}\underline{F}_5\underline{\rho}^* + \underline{\sigma}^{*T}\underline{F}_6\underline{\rho}^* \quad (31)$$

$$\underline{\Gamma} = \underline{\phi}^{*T}\underline{F}_1\underline{r} + \underline{\sigma}^{*T}\underline{F}_2\underline{q} + \underline{\rho}^{*T}\underline{F}_3\underline{p} + \underline{\phi}^{*T}\underline{F}_4\underline{q} + \underline{\phi}^{*T}\underline{F}_5\underline{p} + \underline{\sigma}^{*T}\underline{F}_6\underline{p} \quad (32)$$

$$\underline{\Omega}^T = \underline{r}^T\underline{F}_1\underline{\phi}^* + \underline{q}^T\underline{F}_2\underline{\sigma}^* + \underline{p}^T\underline{F}_3\underline{\rho}^* + \underline{r}^T\underline{F}_4\underline{\sigma}^* + \underline{r}^T\underline{F}_5\underline{\rho}^* + \underline{q}^T\underline{F}_6\underline{\rho}^* \quad (33)$$

$$\underline{\Sigma} = \underline{r}^T\underline{F}_1\underline{r} + \underline{q}^T\underline{F}_2\underline{q} + \underline{p}^T\underline{F}_3\underline{p} + \underline{r}^T\underline{F}_4\underline{q} + \underline{r}^T\underline{F}_5\underline{p} + \underline{q}^T\underline{F}_6\underline{p} \quad (34)$$

where $\underline{F}_1, \ldots, \underline{F}_6$ are constant matrices that depend on structural parameters and the performance index. If, for notational convenience, $\underline{B}_{inv}=\underline{B}^{-1}$, then

$$\underline{F}_1 = \underline{M}^T\underline{B}_{inv}^T\underline{R}\underline{B}_{inv}\underline{M} \quad (35)$$

$$\underline{F}_2 = \underline{C}^T\underline{B}_{inv}^T\underline{R}\underline{B}_{inv}\underline{C} + \underline{Q}_2 \quad (36)$$

$$\underline{F}_3 = \underline{K}^T\underline{B}_{inv}^T\underline{R}\underline{B}_{inv}\underline{K} + \underline{Q}_1 \quad (37)$$

$$\underline{F}_4 = 2\underline{M}^T\underline{B}_{inv}^T\underline{R}\underline{B}_{inv}\underline{C} \quad (38)$$

$$\underline{F}_5 = 2\underline{M}^T\underline{B}_{inv}^T\underline{R}\underline{B}_{inv}\underline{K} \quad (39)$$

$$\underline{F}_6 = 2\underline{C}^T\underline{B}_{inv}^T\underline{R}\underline{B}_{inv}\underline{K} + \underline{Q}_3 \quad (40)$$

Since $\underline{y}$ is independent of time, equation (30) can be written as

$$J_2 = \underline{y}^T\underline{\Lambda}^*\underline{y} + \underline{y}^T\underline{\Gamma}^* + \underline{\Omega}^{*T}\underline{y} + \underline{\Sigma}^* \quad (41)$$

where

$$\underline{\Lambda}^* = \int_0^{t_f}\underline{\Lambda}\,dt \ , \quad \underline{\Gamma}^* = \int_0^{t_f}\underline{\Gamma}\,dt \quad (42),(43)$$

$$\underline{\Omega}^* = \int_0^{t_f}\underline{\Omega}\,dt \ , \quad \underline{\Sigma}^* = \int_0^{t_f}\underline{\Sigma}\,dt \quad (44),(45)$$

Since $J = J_1 + J_2$ is quadratic in terms of $\underline{y}$, the necessary and sufficient condition for global minimum J, determined from

$$\frac{dJ}{d\underline{y}} = \underline{0} \ , \quad (46)$$

is

$$[\underline{\Lambda}^* + \underline{\Lambda}^{*T} + 2\underline{Z}^T\underline{H}\underline{Z}]\underline{y} = -\underline{\Gamma}^* - \underline{\Omega}^* \quad (47)$$

Equation (47) represents a system of linear algebraic equations with the number of equations equal to the number of unknown variables, *i.e.*, the elements of $\underline{y}$. It can be solved using any of a variety of linear equation solvers, such as Gaussian elimination routines. In solving this equation for $\underline{y}$, the integrals of equations (42) - (44) must be evaluated. This can be done numerically or analytically. The integrals have been evaluated in closed-form. The fact that the integral tables can be evaluated analytically makes the Fourier-based approach an integration-free method. As a result, the computational cost is independent of the length of time of the trajectory, making the approach substantially more efficient than standard approaches (except possibly for the case of exceedingly small time intervals.)

An important feature of equation (47) is that the coefficient matrix of $\underline{y}$ is independent of initial conditions. The integrals $\underline{\Lambda}^*$ are independent of initial conditions (whereas the integrals $\underline{\Gamma}^*$ and $\underline{\Omega}^*$ are functions of initial conditions, terminal time, and system parameters.) Thus, for the same optimal control problem with different initial conditions, the coefficient matrix remains the same; only the right-hand side constant vector needs to be recomputed. As a result, numerical algorithms such as LQ decomposition (and linear algebraic equation solvers based on matrix inversion) are particularly efficient for recalculation of $\underline{y}$ for different initial conditions.

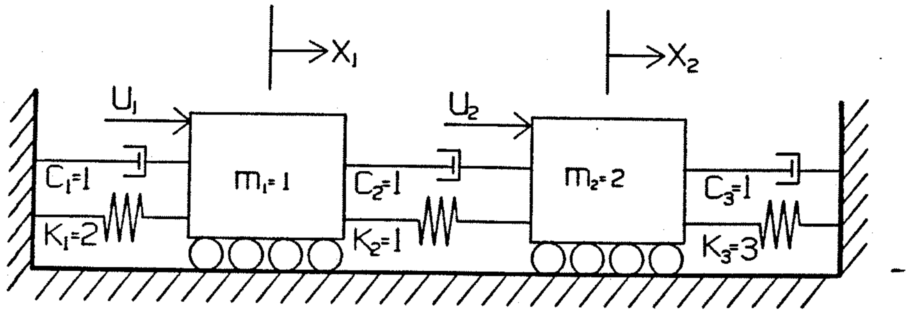

Figure 1. Two Degree-of-Freedom Mechanical System.

EXAMPLES

Example 1

Problem Statement: Consider the linear, two degree-of-freedom, mechanical system shown in Figure 1. The displacements x_1 and x_2 are measured with respect to the equilibrium positions of the masses. For this system, the equation of motion can be written in matrix form as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \ddot{\underline{x}} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \dot{\underline{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \underline{x} = \underline{u}$$

with initial conditions $\underline{x}(0) = [\,1 \quad 1\,]^T$ and $\dot{\underline{x}}(0) = [\,1 \quad 1\,]^T$, where $\underline{x} = [\,x_1 \quad x_2\,]^T$.

The problem is to determine the time history of the control $\underline{u} = [\,u_1 \quad u_2\,]^T$ that minimizes the performance index, equation (2), where

$$\underline{H} = 100\underline{I}_{4x4} \quad , \quad \underline{R} = \underline{I}_{2x2}$$

where $\underline{I}_{nxn}$ is an $n\,x\,n$ identity matrix and

$$\underline{Q}_1 = \underline{Q}_2 = \underline{Q}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution: This problem can be solved by standard linear optimal control methods employing the Hamilton-Jacobi approach via the Riccati equation. Alternatively, the problem can be solved using the proposed Fourier-based approach. Here K is set to 1, *i.e.*, the crudest approximation involving a one-term Fourier-type series is employed. The system of linear algebraic equations (47) can be solved for $\underline{y}$ from which the configuration vector, its rates, and the control vector can be determined.

For this example, the Riccati equations were solved using a fourth-order Runge-Kutta method with a time step of 0.01 sec. Running in Turbo Pascal (Version 3.02A) on an IBM PC/XT with an 8087 co-processor, the computational time was 76 sec. In comparison, the Fourier-based approach required less than 3 sec to establish and solve the linear algebraic equations for the vector of free variables, $\underline{y}$, using a Gauss-Jordan routine.

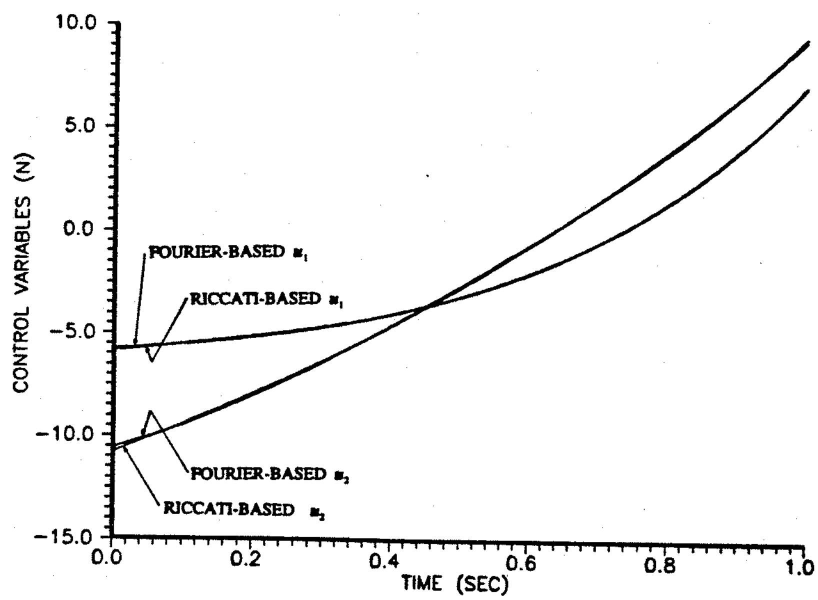

Figure 2. Riccati-Based and Fourier-Based Optimal Solutions of the Control Variables for Example 1.

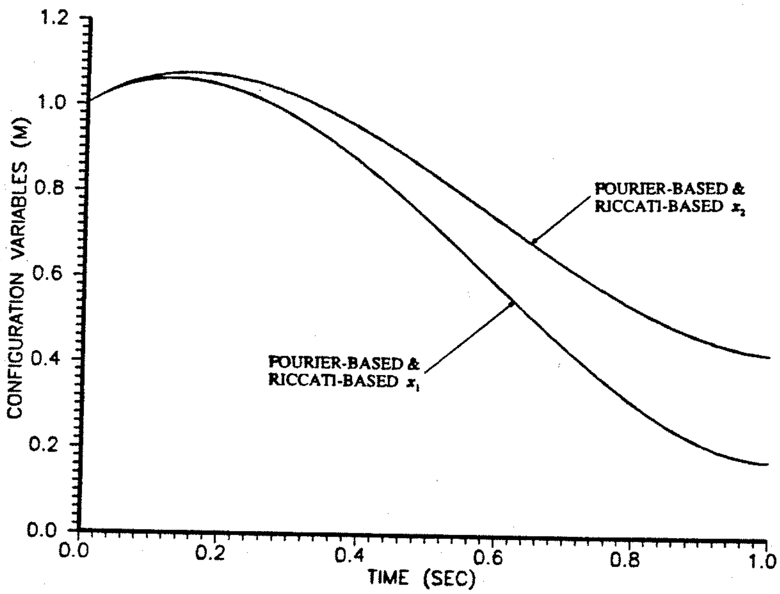

Figure 3. Riccati-Based and Fourier-Based Optimal Solutions of the Configuration Variables for Example 1.

The time responses of the control variables, u_1 and u_2, and the displacements, x_1 and x_2, are shown in Figures 2 and 3, respectively. The results show that the Fourier-based optimal trajectories determined using only a one-term Fourier-type series agree quite well with the optimal trajectories from the Riccati solution.

Further tests were conducted to examine the numerical robustness of the Riccati and Fourier-based methods when high penality was placed on the terminal configuration. The results indicate that the time step for integration of the Riccati equations must be decreased to avoid numerical instability as the penalty is increased, making the standard approach computationally intensive. In contrast, the Fourier-based approach can be applied to systems with highly penalized terminal states without sacrificing the method's computational simplicity.

__Example 2__

This example is identical to Example 1 except that only one control variable is available, *i.e.*, $u_1 = 0$ and $u_2 = u$, and there is no penalty on velocity in the performance index,

$$J = 100[x_1^2(1)+x_2^2(1)] + \int_0^1 (x_1^2+x_2^2+u^2)\,dt$$

To solve this problem using the Fourier-based approach, an artificial control variable, u_1, is introduced as follows:

$$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\ddot{\underline{x}} + \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\dot{\underline{x}} + \begin{bmatrix}2 & -1\\ -1 & 3\end{bmatrix}\underline{x} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

with

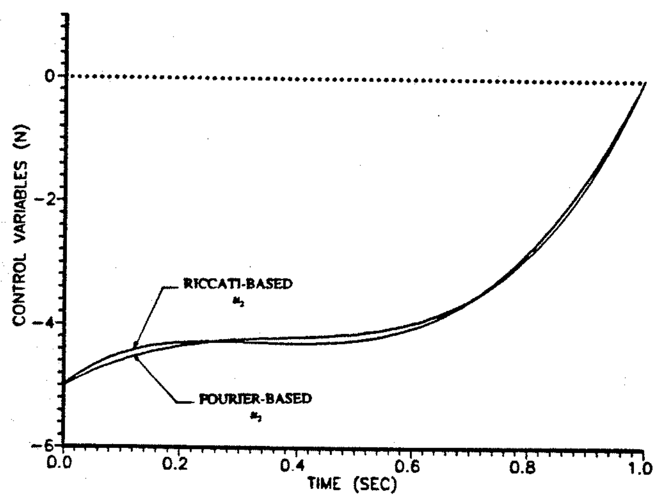

Figure 4. Riccati-Based and Fourier-Based Optimal Solutions of the Control Variable for Example 2.

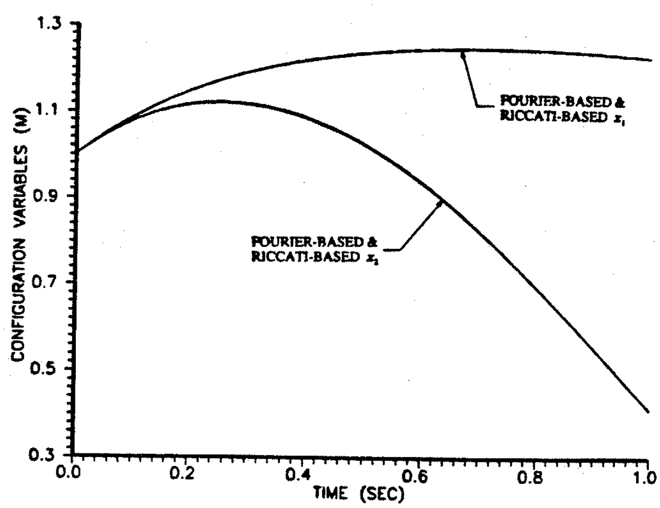

Figure 5. Riccati-Based and Fourier-Based Optimal Solutions of the Configuration Variables for Example 2.

$$J=100[x_1^2(1)+x_2^2(1)]+\int_0^1(x_1^2+x_2^2+u_2^2+Ru_1^2)\,dt$$

A large magnitude of the weighting constant R will ensure that the artificial control u_1 will be small. Here, a value of R = 10,000 was selected (arbitrarily) and a two-term Fourier-type series was used.

The results of the control variable and displacement time histories are shown in Figures 4 and 5, respectively. Here, the configuration variables determined from the Fourier-based and Riccati methods are in agreement, whereas the control variables show a slight discrepancy, which is due to differences in the configuration variable rates.

DISCUSSION

The approach presented in this paper applies to unconstrained linear optimal control problems with quadratic performance indices. It is applicable to high dimension systems and to systems with highly penalized terminal configuration variables (and rates). Unlike variational approaches, the approach does not require integration of differential equations. The optimal (or more correctly, near optimal) solution is obtained by solving a system of linear algebraic equations for free, time-independent parameters. As a result, the approach is computationally very efficient.

By modifying equations (16), (21), and (22), it is possible to apply the method to systems with fixed terminal conditions. For example, if the terminal configuration variable x_{if} is given, then the term $\rho_2 x_{if}$ is known, and one element of equation (16) can be written as:

$$x_i(t) = [p_i+\rho_2 x_{if}]+[\rho_1\ \rho_3\ \rho_4\ \alpha_1\ldots\alpha_K\ \beta_1 .\ \ [x_{i0}\ \dot{x}_{if}\ \ddot{x}_{if}\ a_{i1}\ldots a_{iK}\ b_{i1}\ldots b_{iK}]^T \quad (48)$$

Note that ρ_2 and x_{if} have been removed from equations (14) and (15), respectively, since x_{if} is no longer a free variable.

In the same fashion, problems with fixed terminal configuration variable rates (*i.e.*, $\dot{x}_{if}$ and/or $\ddot{x}_{if}$ known) and problems with fixed initial configuration variable "accelerations" (*i.e.*, $\ddot{x}_{i0}$) can be handled. In practice, once the fixed boundary conditions are identified, the corresponding rows and columns can be extracted from the coefficient matrix of $\underline{y}$ in equation (47) and the contributions of the extracted columns can be subtracted from the corresponding elements of the right-hand side column vector. Using this technique, the same computer routines can be used to handle problems with both fixed and free boundary conditions, eliminating any additional analytical work. Furthermore, problems with linear equality constraints on the boundary conditions (*e.g.*, $x_1(t_f)+x_2(t_f)=1$) can be handled in the same manner.

CONCLUSIONS

The formulation introduced in this paper represents a computationally efficient alternative to standard approaches for the solution of optimal trajectories of linear structural systems with quadratic performance indices. The approach relies on a Fourier-based approximation of the configuration vector. The performance index, which initially is written as a quadratic functional (*i.e.*, a function of configuration and control variables which themselves are functions of time), is converted into a quadratic function (*i.e.*, a function of time-independent parameters of the Fourier-based approach). By differentiating this quadratic function with respect to the free parameters, the necessary and sufficient condition of optimality is derived as a system of linear algebraic equations which can readily be solved.

A major advantage of this approach is its computational efficiency, due to the fact that the optimal configuration and control solutions are found from a system of linear algebraic equations. In contrast, standard optimal control methods typically require the integration of differential Riccati equations. A second advantage of the approach is that it can handle both free and fixed boundary conditions on the configuration vector, in contrast to linear optimal control methods that cannot account directly for boundary condition requirements. Finally, by utilizing quadratic programming techniques, the approach can incorporate linear constraints on configuration and/or control vectors, although this feature was not explored in this paper.

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