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Fourier-Based State Parameterization for Linear Quadratic Optimal Control

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ABSTRACT

This paper considers the optimal control of linear time-invariant dynamical systems with quadratic performance indices. The proposed approach approximates each state variable of a state-space model by the sum of a third order polynomial and a finite term Fourier-type series. In contrast to standard linear optimal control approaches which typically require the solution of Riccati equations, the method adopted here is a near optimal approach in which the necessary and sufficient condition of optimality is derived as a system of linear algebraic equations. These equations can be solved directly by a method such as Gaussian elimination, making the approach computationally efficient.

INTRODUCTION

The optimal control of linear, time-invariant, lumped-parameter, dynamical systems is the subject of much theoretical and practical interest, and is well covered in many textbooks such as (Athans and Falb, 1966; Kirk, 1970; Sage and White, 1977; Lewis, 1986). In the literature, these systems are typically represented by state-space models involving linear, first-order differential equations with constant coefficients. One of the most common approaches for determining the optimal control of linear dynamical systems with quadratic performance indices is the Hamilton-Jacobi approach. Mathematically, it is a variational method which in general requires the solution of a matrix differential Riccati equation with a terminal condition. Various algorithms have been proposed to solve this type of equation; an extensive reference list can be found in (Ramesh, *et al.*, 1987). These algorithms generally suffer from computational "bottlenecks" in solving for the optimal control of high order systems.

In contrast to variational methods, mathematical programming techniques represent a distinct approach toward the solution of (linear and nonlinear) optimal control problems. In general, these techniques convert an optimal control problem into an algebraic optimization problem. A survey of work done prior to 1970 can be found in (Tabak, 1970). A more recent survey can be found in (Kraft, 1980). Theoretical aspects of determining the optimal control via mathematical programming are also covered in (Canon, *et al.*, 1970; Tabak and Kuo, 1971).

A direct application of mathematical programming is to discretize the state equations using a finite difference method. A linear or nonlinear programming algorithm can then be used to determine the values of state and control variables at every time interval such that a performance index is minimized. A difficulty with this approach is that the finite difference approximation leads to a system of algebraic equations which is typically of very large order. As a result, the optimization is computationally intensive and can pose serious problems in obtaining a realistic solution.

Modified approaches involving mathematical programming have been proposed. In (Hicks and Ray, 1971) and (Sirisena and Tan, 1974) the control variables are represented by the sum of known basis functions. Mathematical programming algorithms are then used to determine the optimal values of the coefficients of the basis functions that minimize a performance index. To evaluate the performance index, such control parameterization methods require the integration of the state equations which is usually time consuming and sensitive to numerical errors. Furthermore, constraints on terminal states (*e.g.*, fixed terminal conditions) are not easily satisfied.

Mathematical programming approaches based on state parameterization have been described (Johnson 1969; Nair, 1969; Yen and Nagurka, 1987). In these approaches, state trajectory parameters of dynamical systems are adjusted by mathematical programming. For example, Yen and Nagurka (1987) represent the time history of each generalized coordinate by an auxiliary polynomial and a finite-term Fourier-type series. The free variables, such as the (free) coefficients of the polynomial and the Fourier-type series, are adjusted by a nonlinear programming method such that the performance index is minimized. The effectiveness of this technique has been demonstrated by simulation studies (Yen and Nagurka, 1987). A challenge of state parameterization relates to the problem of trajectory inadmissibility, *i.e.*, due to constraints on the control structure an arbitrary representation of the state trajectory may not be achievable.

Finally, state and control parameterization approaches have been suggested. In (Vlassenbroeck and Van Doreen, 1988) the state and control variables are both expanded in Chebyshev series and an

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algorithm is provided for approximating the system dynamics, boundary conditions and performance index. Here the Chebyshev coefficients are the free variables of the algebraic optimization problem. Although an advantage of this approach is that it can handle linear as well as nonlinear problems, the drawback is the tedious analytical formulation required for different optimal control problems (including unconstrained problems). The development of a general computational tool based on this method is formidable.

Of the different mathematical programming approaches, state parameterization offers two major advantages. First, boundary condition requirements on state variables can be handled directly. Second, if the trajectory inadmissibility problem can be overcome, the state equations can be used as algebraic equations and the performance index can be evaluated efficiently. As a result, state parameterization promises significant computational advantages relative to other approaches.

The research reported here specializes the Fourier-based state parameterization approach of (Yen and Nagurka, 1987) to linear, time-invariant, dynamical systems with quadratic performance indices. The method exploits the linearity of the system model and the quadratic nature of the performance index to guarantee identification of a global minimum, while being computationally very efficient.

METHODOLOGY

The behavior of a linear, time-invariant dynamical system is governed by the state-space model:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) \tag{1}$$

with initial conditions $\underline{x}(0) = \underline{x}_0$ where $\underline{x}$ is an $N \times 1$ state vector (*i.e.*, a column vector of N state variables), $\underline{u}$ is an $L \times 1$ control vector, $\underline{A}$ is an $N \times N$ system matrix, and $\underline{B}$ is an $N \times L$ control matrix. In this paper, it is assumed that $L \leq N$, *i.e.*, the number of control variables is less than or equal to the number of state variables.

The design goal is to find the optimal control $\underline{u}(t)$ in the time interval $[0, t_f]$ such that the quadratic performance index

$$J = \underline{x}^T(t_f)\underline{H}\underline{x}(t_f) + \int_0^{t_f}[\underline{x}^T\underline{Q}\underline{x} + \underline{u}^T\underline{R}\underline{u}]dt \tag{2}$$

is minimized. Here, $\underline{Q}$ and $\underline{H}$ are real, symmetric, positive, semidefinite matrices and $\underline{R}$ is a real, symmetric, positive definite matrix. (T denotes transpose.) It is assumed that the state and control vectors are not bounded, the terminal state $\underline{x}(t_f)$ is free, and the terminal time t_f is fixed.

Linear Systems with Nonsingular Control Matrix

This subsection considers the case $L = N$, *i.e.*, the state and control vectors have the same dimension, and hence $\underline{B}$ is square. Furthermore, it is assumed that $\underline{B}$ is nonsingular. The following subsection addresses the case $L<N$.

The basic idea of the proposed approach is to approximate each state variable by the sum of an auxiliary polynomial and a finite term Fourier-type series, *i.e.*, for $i = 1, \ldots, N$,

$$x_i(t) = d_i(t) + \sum_{k=1}^{K} a_{ik} \cos\frac{2k\pi t}{t_f} + \sum_{k=1}^{K} b_{ik} \sin\frac{2k\pi t}{t_f} \tag{3}$$

where K is the number of terms included in the Fourier-type series and d_i is a third-order polynomial in time

$$d_i(t) = d_{i0} + d_{i1}t + d_{i2}t^2 + d_{i3}t^3 \tag{4}$$

The four coefficients of this auxiliary polynomial can be written in terms of four boundary conditions, *i.e.*, initial conditions $x_i(0)$ and $\dot{x}_i(0)$ and terminal conditions $x_i(t_f)$ and $\dot{x}_i(t_f)$. Explicit expressions for these coefficients are given in Appendix A.

Equation (3) can be rearranged and presented in the form

$$x_i(t) = p_i + \rho_1\dot{x}_{i0} + \rho_2 x_{if} + \rho_3\dot{x}_{if} + \sum_{k=1}^{K}\alpha_k a_{ik} + \sum_{k=1}^{K}\beta_k b_{ik} \tag{5}$$

where $x_{i0} = x_i(0)$, $x_{if} = x_i(t_f)$, and similarly for the corresponding time derivatives, and where

$$p_i = (1-3\tau^2+2\tau^3)x_{i0} \tag{6}$$

$$\rho_1 = (\tau-2\tau^2+\tau^3)t_f \tag{7}$$

$$\rho_2 = 3\tau^2 - 2\tau^3 \quad , \quad \rho_3 = (-\tau^2 + \tau^3)t_f \qquad (8),(9)$$

$$\alpha_k = \cos(2k\pi\tau) - 1 \tag{10}$$

$$\beta_k = \sin(2k\pi\tau) - 2k\pi\tau(1 - 3\tau + 2\tau^2) \tag{11}$$

where $\tau = (t/t_f)$.

Since the terminal time t_f is assumed known, the parameters defined in equations (7)-(11) are functions of time only, and are state independent. Furthermore, since the initial condition x_{i0} is given, p_i in equation (6) is a known function of time.

From equation (5), the state variable $x_i(t)$ can be written in compact form as

$$x_i(t) = p_i + \underline{\rho}^T\underline{y}_i \tag{12}$$

where

$$\underline{\rho}^T = [\ \rho_1\ \ \rho_2\ \ \rho_3\ \ \alpha_1\ \ldots\ \alpha_K\ \ \beta_1\ \ldots\ \beta_K\ ] \tag{13}$$

and

$$\underline{y}_i = [\ \dot{x}_{i0}\ \ x_{if}\ \ \dot{x}_{if}\ \ a_{i1}\ \ldots\ a_{iK}\ \ b_{i1}\ \ldots\ b_{iK}\ ]^T \tag{14}$$

are vectors of dimension $m = 3 + 2K$.

The N state variables can be written in terms of state vector $\underline{x}(t)$, *i.e.*,

$$\underline{x}(t) = \underline{p}(t) + \underline{\rho}^*(t)\underline{y} \tag{15}$$

where

$$\underline{x}(t) = [x_1(t)\ \ x_2(t)\ \ .\ .\ .\ \ x_N(t)]^T \tag{16}$$

$$\underline{p}(t) = [p_1(t)\ \ p_2(t)\ \ .\ .\ .\ \ p_N(t)]^T \tag{17}$$

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_N \end{bmatrix} \quad , \quad \underline{\rho}^* = \begin{bmatrix} \underline{\rho}^T & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{\rho}^T & \cdots & \underline{0} \\ \vdots & & \ddots & \vdots \\ \underline{0} & \cdots & & \underline{\rho}^T \end{bmatrix} \qquad (18),(19)$$

Note that $\underline{y}$ is a column vector of dimension Nm and that $\underline{\rho}^*$ is a matrix of dimension $N \times Nm$. Similarly, the time derivative of the state vector,

$$\underline{\dot{x}}(t) = \underline{q}(t) + \underline{\sigma}^*(t)\underline{y} \qquad (20)$$

can be introduced, where

$$\underline{q} = \underline{\dot{p}} \quad , \quad \underline{\sigma}^* = \underline{\dot{\rho}}^* \qquad (21),(22)$$

Since $\underline{x}(t)$ and $\underline{\dot{x}}(t)$ are known functions of $\underline{y}$, the control vector $\underline{u}(t)$ can be expressed from equation (1) as a function of $\underline{y}$. Ultimately, the interest is to express the performance index as a function of $\underline{y}$. Toward this end, the performance index of equation (2) is decomposed into two parts:

$$J = J_1 + J_2 \qquad (23)$$

where J_1 is the cost associated with the terminal state and J_2 is the cost associated with the trajectory. The terminal state can be written as a linear transformation of $\underline{y}$, *i.e.,*

$$\underline{x}(t_f) = \underline{Z}\underline{y} \qquad (24)$$

where $\underline{Z}$ is a $N \times mN$ matrix with elements 1 and 0, specified according to

$$z_{ij} = \begin{cases} 1 \ , & j=(i-1)m+1 \quad \text{for } i=1,\dots,N \\ 0 \ , & \text{otherwise} \end{cases} \qquad (25)$$

From equation (24), the cost J_1 is

$$J_1 = \underline{x}^T(t_f)\underline{H}\underline{x}(t_f) = \underline{y}^T\underline{Z}^T\underline{H}\underline{Z}\underline{y} \qquad (26)$$

From equation (1) (for the control vector) and equations (15) and (20) (for the state vector and its rate), the cost J_2 is

$$J_2 = \int_0^{t_f}[\underline{x}^T\underline{Q}\underline{x}+\underline{u}^T\underline{R}\underline{u}]dt = \int_0^{t_f}[\underline{y}^T\underline{\Lambda}\underline{y}+\underline{y}^T\underline{\Gamma}+\underline{\Sigma}]\,dt \qquad (27)$$

In equation (27)

$$\underline{\Lambda} = \underline{\rho}^{*T}\underline{F}_1\underline{\rho}^* + \underline{\sigma}^{*T}\underline{F}_2\underline{\sigma}^* + \underline{\sigma}^{*T}\underline{F}_3\underline{\rho}^* \qquad (28)$$

$$\underline{\Gamma}=\underline{\rho}^{*T}(\underline{F}_1+\underline{F}_1^T)\underline{p}+\underline{\sigma}^{*T}(\underline{F}_2+\underline{F}_2^T)\underline{q}+\underline{\sigma}^{*T}\underline{F}_3\underline{p}+\underline{\rho}^{*T}\underline{F}_3\underline{q} \qquad (29)$$

$$\underline{\Sigma} = \underline{p}^T\underline{F}_1\underline{p} + \underline{q}^T\underline{F}_2\underline{q} + \underline{q}^T\underline{F}_3\underline{p} \qquad (30)$$

where $\underline{F}_1$, $\underline{F}_2$, and $\underline{F}_3$ are constant matrices that depend on system parameters and the performance index.

$$\underline{F}_1 = \underline{Q} + \underline{C}^T\underline{R}\underline{C} \qquad (31)$$

$$\underline{F}_2 = (\underline{B}^{-1})^T\underline{R}\underline{B}^{-1} \quad , \quad \underline{F}_3 = 2(\underline{B}^{-1})^T\underline{R}\underline{C} \qquad (32),(33)$$

where

$$\underline{C} = -\underline{B}^{-1}\underline{A} \qquad (34)$$

Since $\underline{y}$ is independent of time, equation (27) can be written as

$$J_2 = \underline{y}^T\underline{\Lambda}^*\underline{y} + \underline{y}^T\underline{\Gamma}^* + \underline{\Sigma}^* \qquad (35)$$

where

$$\underline{\Lambda}^* = \int_0^{t_f}\underline{\Lambda}\,dt \quad , \quad \underline{\Gamma}^* = \int_0^{t_f}\underline{\Gamma}\,dt \qquad (36),(37)$$

$$\underline{\Sigma}^* = \int_0^{t_f}\underline{\Sigma}\,dt \qquad (38)$$

Since $J = J_1 + J_2$ is quadratic in terms of $\underline{y}$, the necessary and sufficient condition for global minimum J, determined from

$$\frac{dJ}{d\underline{y}} = \underline{0} \ , \qquad (39)$$

is

$$[\underline{\Lambda}^* + \underline{\Lambda}^{*T} + 2\underline{Z}^T\underline{H}\underline{Z}]\underline{y} = -\underline{\Gamma}^* \qquad (40)$$

Equation (40) represents a system of linear algebraic equations with the number of equations equal to the number of unknown variables, *i.e.,* the elements of $\underline{y}$. It can be solved using any of a variety of linear equation solvers, such as Gaussian elimination routines. In solving this equation for $\underline{y}$, the integrals of equations (36) and (37) must be evaluated. This can be done numerically or analytically. The integrals have been evaluated in closed-form; a sample integral table is shown in Appendix B.

An important feature of equation (40) is that the coefficient matrix of $\underline{y}$ is independent of initial conditions. The integrals $\underline{\Lambda}^*$ are independent of initial conditions (whereas the integrals $\underline{\Gamma}^*$ are functions of initial conditions, terminal time, and system parameters.) Thus, for the same optimal control problem with different initial conditions, the coefficient matrix remains the same; only the right-hand side constant vector needs to be recomputed. As a result, numerical algorithms such as LU decomposition (and linear algebraic equation solvers based on matrix inversion) are particularly efficient for recalculation of $\underline{y}$ for different initial conditions.

General Linear Systems

The approach presented above is applicable only for systems with square and invertible control matrices. This subsection generalizes the Fourier-based approach to the more common case of general linear systems which have fewer control variables than state variables. The dynamical system of interest is again the linear structure described by equation (1). In this case, the control matrix, $\underline{B}$, is an $N \times L$ matrix where the number of state variables, N, is greater than the number of control variables, L.

Equation (1) can be written as

$$\underline{\dot{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}'\underline{u}'(t) \qquad (41)$$

where

$$\underline{B}' = \underline{B}'_{(N\times N)} = \begin{bmatrix} \underline{I}_{(I\times I)} & \vdots \\ & \vdots \underline{B}_{(N\times L)} \\ \underline{0}_{(L\times I)} & \vdots \end{bmatrix} \tag{42}$$

and

$$\underline{u}' = \underline{u}'_{(N\times 1)} = \begin{bmatrix} \underline{u}^*_{(I\times 1)} \\ \\ \underline{u}_{(L\times 1)} \end{bmatrix} \tag{43}$$

where $I = N - L$ and the subscripts in the parentheses represent the dimensions of the matrices. By introducing the artificial control vector, $\underline{u}^*$, the new control matrix, $\underline{B}'$, can be inverted enabling the calculation of the control, $\underline{u}'$, for any given trajectory, *i.e.*, premultiplying equation (41) by $(\underline{B}')^{-1}$ gives

$$\underline{A}'\underline{x}(t) + \underline{B}''\dot{\underline{x}}(t) = \underline{u}'(t) \tag{44}$$

where

$$\underline{A}' = -(\underline{B}')^{-1}\underline{A} \quad , \quad \underline{B}'' = (\underline{B}')^{-1} \tag{45),(46}$$

The new constant coefficient matrices can be partitioned, as follows.

$$\underline{A}' = \begin{bmatrix} \underline{A}^*_{(I\times N)} \\ \\ \underline{A}^{**}_{(L\times N)} \end{bmatrix} , \quad \underline{B}'' = \begin{bmatrix} \underline{B}^*_{(I\times N)} \\ \\ \underline{B}^{**}_{(L\times N)} \end{bmatrix} \tag{47),(48}$$

From equation (44) the artificial control vector $\underline{u}^*$ can thus be written as

$$\underline{A}^*\underline{x}(t) + \underline{B}^*\dot{\underline{x}}(t) = \underline{u}^*(t) \tag{49}$$

In actuality, the artificial control does not exist, and thus it is required that

$$\underline{A}^*\underline{x}(t) + \underline{B}^*\dot{\underline{x}}(t) = \underline{0} \tag{50}$$

This indicates that only trajectories satisfying equation (50) are admissible for dynamical systems described by equation (1). In other words, the system of N state variables possesses L "active" state variables. Given trajectories of any L of the N state variables, the trajectories of the remaining state variables can be determined uniquely from equation (50), with all trajectories being admissible.

Due to the approximation of the Fourier-based approach, equation (50) can not be satisfied exactly. However, by minimizing, in a least squares sense, the contribution of the artificial control variables, an equation similar to equation (50) describing linear coupling between state variables can be derived. Thus, there are two simultaneous objectives. One objective is to generate the near optimal trajectories; the second objective is to minimize, in the least squares sense, the contribution of the artificial control variables.

A performance index, J^*, is proposed to represent the contribution of the artificial control variables.

$$J^* = \int_0^{t_f} \sum_{i=1}^{I} (u_i^*)^2 dt \tag{51}$$

There are N vectors representing the free Fourier-based variables, $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_N$. I of these N vectors, *i.e.*, $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_I$, are adjusted in such a way that J^* is minimized. Setting the first derivative of the performance index equal to zero

$$\frac{dJ^*}{d\underline{y}_i} = \underline{0} \qquad for \quad i = 1,2,\ldots,I \tag{52}$$

gives a set of Im equations in the Nm unknowns ($\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_N$ where each $\underline{y}_i$ has m elements.) Equation (52) can be written as

$$\underline{y}_A = \underline{D}_1\underline{y}_B + \underline{D}_2 \tag{53}$$

where $\underline{y}_A$ and $\underline{y}_B$ are partitioned vectors of $\underline{y}$ according to

$$\underline{y} = \begin{bmatrix} \underline{y}_A \\ \underline{y}_B \end{bmatrix} \tag{54}$$

with

$$\underline{y}_A = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{y}_I \end{bmatrix} , \quad \underline{y}_B = \begin{bmatrix} \underline{y}_{I+1} \\ \underline{y}_{I+2} \\ \cdot \\ \cdot \\ \cdot \\ \underline{y}_N \end{bmatrix} \tag{55),(56}$$

Equation (53) represents the coupling between the state variables that minimizes the effect of the artificial control variables on the trajectories.

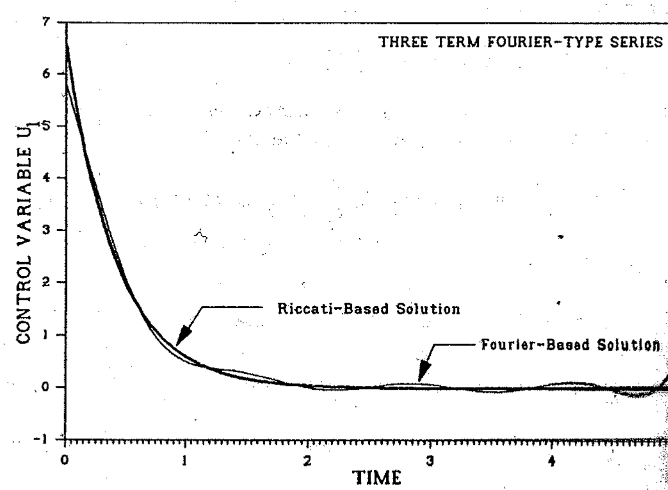

Figure 1a. Control Variable u_1 History for Example 1 (With Three Term Fourier-Type Series)

The performance index, $J = J_1 + J_2$, can be written in terms of $\underline{y}$ according to equations (26) and (35). In view of equation (54),

$$J = J(\underline{y}) = J(\underline{y}_A, \underline{y}_B) \qquad (57)$$

which, from equation (53), can be written as

$$J = J(\underline{y}_B) \qquad (58)$$

The necessary and sufficient condition of optimality can then be expressed as

$$\frac{dJ}{d\underline{y}_B} = \underline{0} \ , \qquad (59)$$

which represents Lm algebraic equations that can be solved to determine $\underline{y}_B$. The remaining Fourier-based variables, $\underline{y}_A$, can then be computed from equation (53).

EXAMPLES

Example 1: Linear System with Nonsingular Control Matrix

This example, which is an adapted version of an example problem from (Kirk, 1970, Example 5.2-2), considers a linear dynamical system with two state and control variables

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u} \qquad (60)$$

with initial conditions $\underline{x}(0) = [-4 \quad 4]^T$, where $\underline{x} = [x_1 \quad x_2]^T$. The problem is to determine the time history of the control vector $\underline{u} = [u_1 \quad u_2]^T$ that minimizes the performance index of equation (2) where

$$\underline{H}=\underline{0} \ , \ \underline{Q}=diag[2,1] \ , \ \underline{R}=diag[0.5,0.5] \qquad (61)$$

and where the terminal time $t_f = 5$.

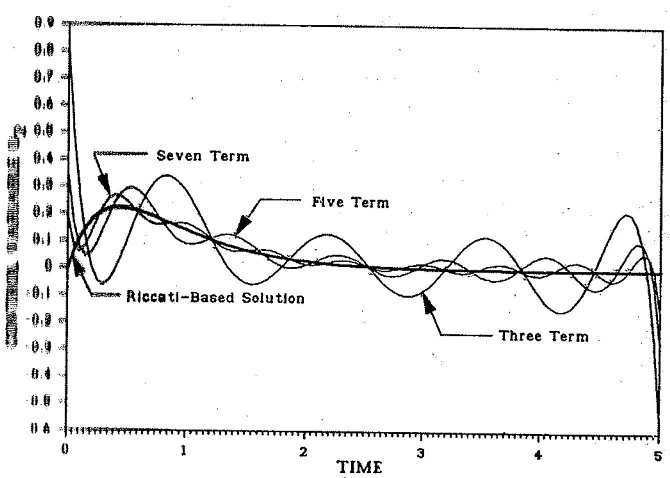

Figure 1b. Control Variable u_2 History for Example 1 (With Three, Five, & Seven Term Fourier-Type Series)

Table 1: Value of Performance Index as a Function of Number of Fourier-Based Terms (K) for Example 1.

$J_{Riccati} = 6.5830$

Number of Terms, K	$J_{Fourier\text{-}Based}$	Percent Error
1	7.4145	12.63
2	6.7402	2.39
3	6.6245	0.630
4	6.5968	0.210
5	6.5885	0.084
6	6.5855	0.038
7	6.5843	0.020
8	6.5836	0.009
9	6.5834	0.006

The optimal solution was determined by solving the Riccati equation and by employing the Fourier-based approach. The time history of the first control variable, u_1, is shown in Figure 1a. The Fourier-based solution with three terms (*i.e.,* $K = 3$) compares quite favorably with the Riccati solution. Figure 1b shows the time history of the second control variable, u_2, from the Riccati solution and from the Fourier-based approach with three, five and seven terms. There is an apparent deviation from the Riccati solution at the endpoints. Furthermore, the Riccati solution shows that both control variables approach steady-state values of zero after $t = 2$, whereas the Fourier-based solutions are characterized by oscillatory behavior.

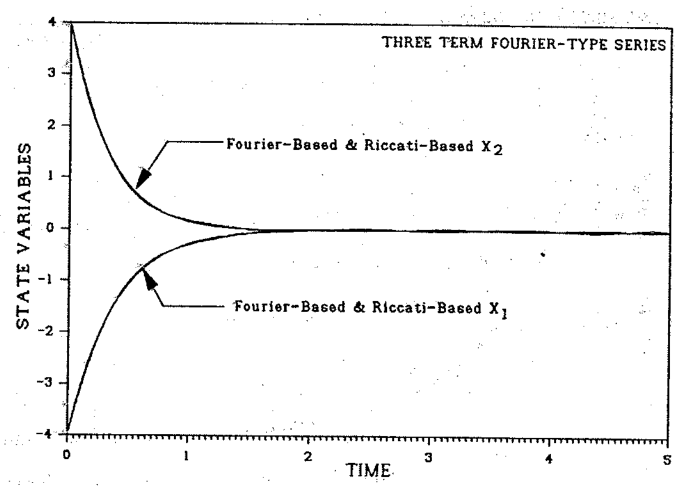

Figure 1c. State Variable History for Example 1 (With Three Term Fourier-Type Series)

Figure 1b suggests that the Fourier-based approach may be lacking in accuracy, especially when few Fourier-type terms are included. However, discrepancies of the control variables are not fully reflective of the quality of the Fourier-based solution, as demonstrated in Table 1 which lists the value of the performance index as a function of the number of Fourier-type terms. With a three term approximation, the Fourier-based solution deviates from the "exact" Riccati solution by 0.63 percent, an inaccuracy well within engineering tolerance. As a further indication of the effectiveness of the proposed approach, the state variable histories match those of the Riccati solution. Figure 1c shows the Riccati-based and three-term Fourier-based state trajectories, $x_1(t)$ and $x_2(t)$.

Example 2: General Linear System

In this example (Kirk, 1970, Example 5.2-2) the model is represented by a single input, second order system, *i.e.*, there are two state variables, x_1 and x_2, and one control variable, u. The system matrix, $\underline{A}$, the terminal state weighting matrix, $\underline{H}$, and the state weighting matrix, $\underline{Q}$, are the same as in Example 1. Here, the control matrix is a vector, $\underline{B} = [\,0 \quad 1\,]^T$, and the control weighting matrix is a scalar, $R = 0.5$. As before, the terminal time $t_f = 5$.

Using the approach outlined above for general linear systems, an artificial control variable, u^*, is introduced and a new control, $\underline{B}'$, is formed according to equation (42). Here, $\underline{B}'$, is a 2 x 2 identity matrix. Following the procedure for simultaneous optimization (equations (51) - (59)) the Fourier-based solution can be obtained.

Figures 2a and 2b compare the control variable history from the Riccati and Fourier-based methods with one and three term series, respectively. A slight discrepancy is evident with the one-term Fourier-type series (Figure 2a), whereas the three-term solution (Figure 2b) appears coincident with the Riccati solution. The time history of the corresponding artificial control variable for the one and three-term solutions is plotted in Figure 2c. As expected, the artificial control variable based on three-term Fourier-type series is closer to zero than the artificial variable based on the one-term series. However, the magnitudes are small for both cases and hence the influence of the artificial control variables on the system dynamics is negligible.

DISCUSSION

The Fourier-based state parameterization approach presented in this paper applies to unconstrained linear optimal control problems with quadratic performance indices. The parameters, *i.e.*, the elements of $\underline{y}$, include the free boundary conditions and the coefficients of the Fourier-type series. The optimal control is found by solving for the "optimal" values of these parameters such that a performance index is minimized.

The approach is applicable to high order systems and to systems with highly penalized terminal state variables (and rates). Unlike variational approaches, the approach does not require integration of differential equations. The optimal (or more correctly, near optimal) solution is obtained by solving a system of linear algebraic equations for free, time-independent parameters. As a result, the approach is computationally very efficient.

By modifying equations (15) and (20), it is possible to apply the method to systems with fixed terminal conditions. For example, if the terminal state variable x_{if} is given, then the term $\rho_2 x_{if}$ is known, and one element of equation (15) can be written as:

$$x_i(t) = [p_i + \rho_2 x_{if}] + [\rho_1 \ \ \rho_3 \ \ \rho_4 \ \ \alpha_1 \ldots \alpha_K \ \ \beta_1 \ldots \beta_K] \, [\dot{x}_{i0} \ \ \dot{x}_{if} \ \ a_{i1} \ldots a_{iK} \ \ b_{i1} \ldots b_{iK}]^T \qquad (62)$$

Note that ρ_2 and x_{if} have been removed from equations (13) and (14), respectively, since x_{if} is no longer a free variable.

In the same fashion, problems with fixed initial or terminal state variable rates (*i.e.*, $\dot{x}_{i0}$ or $\dot{x}_{if}$ known) can be handled. In practice, once the fixed boundary conditions are identified, the corresponding rows and columns can be extracted from the coefficient matrix of $\underline{y}$ in equation (40) and the contributions of the extracted columns can be subtracted from the corresponding elements of the right-hand side column vector. Using this technique, the same computer routines can be used to handle problems with both fixed and free boundary conditions, eliminating any additional analytical work. Furthermore, problems with linear equality constraints on the boundary conditions (*e.g.*, $x_1(t_f) + x_2(t_f) = 1$) can be handled in the same manner.

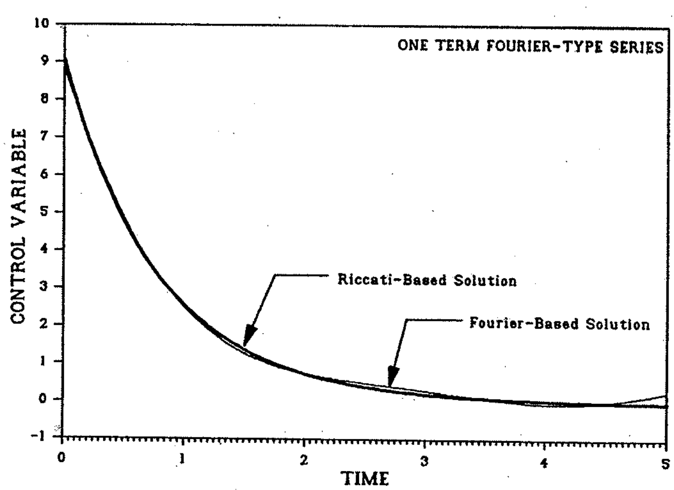

Figure 2a. Control Variable History for Example 2 (With One Term Fourier-Type Series)

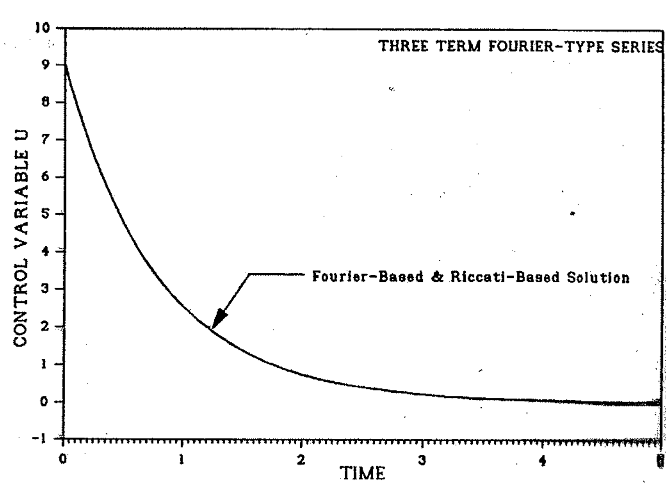

Figure 2b. Control Variable History for Example 2 (With Three Term Fourier-Type Series)

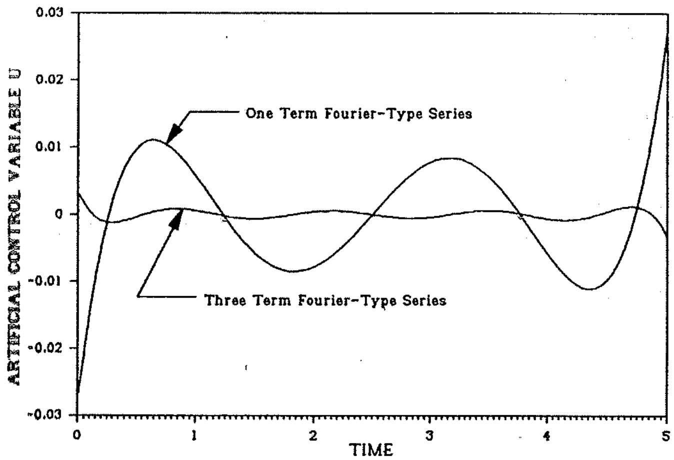

Figure 2c. Artificial Control Variable History for Example 2 (One and Three Term Fourier-Type Series)

CONCLUSIONS

The formulation introduced in this paper represents a computationally efficient alternative to standard approaches for the solution of optimal trajectories of linear time-invariant dynamical systems with quadratic performance indices. The approach relies on a Fourier-based approximation of the state vector. The performance index, which initially is written as a quadratic functional (*i.e.*, a function of state and control variables which themselves are functions of time), is converted into a quadratic function (*i.e.*, a function of time-independent parameters of the Fourier-based approach). By differentiating this quadratic function with respect to the free parameters, the necessary and sufficient condition of optimality is derived as a system of linear algebraic equations which can readily be solved.

A major advantage of this approach relates to the ease with which the optimal state and control solutions can be found, *i.e.*, by solving a system of linear algebraic equations. In contrast, standard optimal control methods typically require the integration of differential Riccati equations. A second advantage of the approach is that it can handle both free and fixed boundary conditions on the state vector, in contrast to linear optimal control methods that cannot account directly for boundary condition requirements. Finally, the approach promises to be a useful tool for linear quadratic regulator design, a feature that will be explored in a subsequent paper.

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APPENDIX A: Auxiliary Polynomial Coefficients

The coefficients of equation (4) are determined from the boundary conditions of x_i, *i.e.*, $x_i(0)$, $\dot{x}_i(0)$, $x_i(t_f)$, and $\dot{x}_i(t_f)$, giving rise to four simultaneous algebraic equations which can be solved for:

$$d_{i0} = x_{i0} - \sum_{k=1}^{K} a_{ik} \qquad (A\text{–}1)$$

$$d_{i1} = \dot{x}_{i0} - \frac{2\pi}{t_f}\sum_{k=1}^{K} k b_{ik} \qquad (A\text{–}2)$$

$$d_{i2} = 3(x_{if}-x_{i0})t_f^{-2} - 2(\dot{x}_{i0}+\dot{x}_{if}-6\pi\sum_{k=1}^{K} k b_{ik})t_f^{-1} \qquad (A\text{–}3)$$

$$d_{i3} = 2(x_{if}-x_{i0})t_f^{-3} + (\dot{x}_{i0}+\dot{x}_{if}-4\pi\sum_{k=1}^{K} k b_{ik})t_f^{-2} \qquad (A\text{–}4)$$

where $x_{i0} = x_i(0)$, $x_{if} = x_i(t_f)$, and similarly for the corresponding time derivatives.

APPENDIX B: Sample Integral Table

Evaluation of $\underline{\Lambda}_1^* = \int_0^{t_f} \underline{\rho}^{*T} \underline{F}_1 \underline{\rho}^* dt$
$(v_k = 2\pi k)$

ω	$\int_0^{t_f} \omega dt$
ρ_1^2	$(\frac{1}{105})t_f^3$
$\rho_1\rho_2$	$(\frac{13}{420})t_f^2$
$\rho_1\rho_3$	$(-\frac{1}{140})t_f^3$
ρ_2^2	$(\frac{13}{35})t_f$
$\rho_2\rho_3$	$(-\frac{11}{210})t_f^{\ 2}$
ρ_3^2	$(\frac{1}{105})t_f^3$
$\rho_1\alpha_k$, $-\rho_3\alpha_k$	$-(\frac{1}{v_k^2} + \frac{1}{12})t_f^2$
$\rho_2\alpha_k$	$(-\frac{1}{2})t_f$
$\rho_1\beta_k$, $\rho_3\beta_k$	$(\frac{6}{v_k^3} - \frac{v_k}{420})t_f^2$
$\rho_2\beta_k$	$-(\frac{12}{v_k^3} + \frac{1}{v_k})t_f + (\frac{3}{140})v_k t_f$
$\alpha_j\beta_k$	0
$\alpha_j\alpha_k$, $j \neq k$	t_f
$\alpha_j\alpha_k$, $j = k$	$(\frac{3}{2})t_f$
$\beta_j\beta_k$, $j \neq k$	$[\frac{1}{210}v_j v_k - 12(\frac{v_j}{v_k^3} + \frac{v_k}{v_j^k})]t_f$
$\beta_j\beta_k$, $j = k$	$[\frac{1}{210}v_j v_k - 12(\frac{v_j}{v_k^3} + \frac{v_k}{v_j^k})]t_f + (\frac{1}{2})t_f$