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Linear Quadratic Optimal Control Via Fourier-Based State Parameterization

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A method for determining the optimal control of unconstrained and linearly constrained linear dynamic systems with quadratic performance indices is presented. The method is based on a modified Fourier series approximation of each state variable that converts the linear quadratic (LQ) problem into a mathematical programming problem. In particular, it is shown that an unconstrained LQ problem can be cast as an unconstrained quadratic programming problem where the necessary condition of optimality is derived as a system of linear algebraic equations. Furthermore, it is shown that a linearly constrained LQ problem can be converted into a general quadratic programming problem. Simulation studies for constrained LQ systems, including a bang-bang control problem, demonstrate that the approach is accurate. The results also indicate that in solving high order unconstrained LQ problems the approach is computationally more efficient and robust than standard methods.

Introduction

The optimal control of linear, lumped parameter, dynamic systems is the subject of much theoretical and practical interest, and is well covered in many textbooks (e.g., Athans and Falb, 1966; Kirk, 1970; Sage and White, 1977; Lewis 1986). Typically, the necessary condition of optimality is formulated as a two-point boundary-value problem (TPBVP) using variational methods. Except in some special cases, the solution of this TPBVP is usually difficult, and in some cases not practical, to obtain.

In contrast to variational methods, trajectory parameterization represents a distinct approach toward the solution of optimal control problems. In general, these techniques approximate the control and/or state vectors by functions with unknown coefficients, thereby converting an optimal control problem into a mathematical programming (MP) problem. A near optimal solution can then be obtained via various well developed optimization algorithms. For example, quadratic programming has been used to solve parameterized linear optimal control problems (Canon and Eaton, 1966; Blum and Fegley, 1968; Jizmagian, 1969; Bosarge and Johnson, 1970; Neuman and Sen, 1973). A survey of work done prior to 1970 can be found in (Tabak, 1970); a later study can be found in (Kraft, 1980). Theoretical aspects of solving optimal control problems via trajectory parameterization are also covered in (Canon et al., 1970; Tabak and Kuo, 1971; Luenberger, 1972; Evtushenko, 1985).

A direct application of trajectory parameterization is to parameterize the control variables. For example, after representing the control variables by a sequence of eigenfunctions with unknown weighting coefficients, a linear or nonlinear programming algorithm can be used to determine the coefficients (i.e., control parameters) such that a performance index is minimized. A difficulty with control parameterization occurs in determining the functional relationship between state variables and control parameters. The process of determining this relationship can be analytically cumbersome and most often requires numerical integration of the state equations that can be computationally intensive and sensitive to numerical errors.

Approaches based on state parameterization have been described (Johnson, 1969; Nair, 1978; Yen and Nagurka, 1988, 1989; Nagurka and Yen, 1990). In these approaches, state trajectory parameters are adjusted by MP algorithms in order to minimize a performance index. For example, Nagurka and Yen (1990) present a nonlinear programming approach for determining the near optimal trajectories of linear and nonlinear dynamic systems. They motivate the use of a fifth-order polynomial appended to a Fourier series to represent each generalized coordinate. This representation is used to convert an optimal control problem into an algebraic optimization problem. The free variables, such as the free coefficients of the polynomial and the Fourier series, are adjusted by a nonlinear programming method to minimize a performance index. Two numerical schemes (the Powell and the Simplex methods) are tested on unconstrained and constrained, linear and nonlinear, and fixed and free terminal time, optimal control problems. The results suggest that the Fourier-based method is accurate for solving such problems, except bang-bang type control problems. A general challenge of state parameterization involves the problem of trajectory inadmissibility, i.e., due to constraints on the control structure an arbitrary representation of the state trajectory may not be achievable.

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Finally, combined state and control parameterization approaches have been suggested. In (Vlassenbroeck and Van Dooren, 1988) both the state and control variables are expanded in Chebyshev series. Although their approach can handle linear as well as nonlinear problems, it requires the approximation of the system dynamics, boundary conditions, and performance index, involves tedious analytical formulation for different optimal control problems, and increases the number of unknown variables of the converted MP problem. In general, the number of free variables is typically higher than the number employed in either state or control parameterization approaches.

Of the different trajectory parameterization approaches, state parameterization offers two major advantages. First, boundary condition requirements on the state variables can be handled directly. Second, if the trajectory inadmissibility problem can be overcome, the state equations can be used as algebraic equations. As a result, the process of determining the functional relationship between the state and control vectors is easier to implement in state parameterization than in control parameterization.

This research is part of a broader effort toward the development of a computational tool for solving optimal control problems via state parameterization. As part of this effort, this paper presents a specialized version of the Fourier-based state parameterization approach (Nagurka and Yen, 1990) for determining the optimal trajectories of linear systems described by state-space models with quadratic performance indices and linear constraints. The approach employs a third-order polynomial appended to a Fourier-type series to represent each state variable. For the unconstrained optimal control problem, a system of linear algebraic equations is derived as the condition of optimality from which the near optimal state and control trajectories can be determined. For the linearly constrained problem, the LQ problem is converted to a quadratic programming problem which can be solved by well developed routines. Example problems demonstrate the high accuracy, computational efficiency, and robustness of the method. Simulation studies for constrained LQ systems are included in two examples, one of which addresses the applicability of a multiple segment (i.e., splined) Fourier-based method for solving bang-bang control problems.

Problem Statement

The behavior of a linear dynamic system is described by the state-space model

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{B}(t)\mathbf{u}(t) \qquad (1)$$

with known initial condition $\mathbf{x}(0)=\mathbf{x}_o$ where $\mathbf{x}$ is an $N\times 1$ state vector, $\mathbf{u}$ is a $J\times 1$ control vector, $\mathbf{A}$ is an $N\times N$ system matrix, and $\mathbf{B}$ is an $N\times J$ control influence matrix. For now, it is assumed that $J=N$ and $\mathbf{B}$ is invertible, implying that every state variable can be "actively" controlled. These assumptions will be relaxed later.

The design goal is to find the control $\mathbf{u}(t)$ and the corresponding state trajectory $\mathbf{x}(t)$ in the time interval $[0, T]$ that minimizes the quadratic performance index, L,

$$L=L_1+L_2 \qquad (2)$$

where L_1 is the cost associated with the terminal state

$$L_1=\mathbf{x}^T(T)\mathbf{H}\mathbf{x}(T)+\mathbf{h}^T\mathbf{x}(T) \qquad (3)$$

and L_2 is the cost associated with the trajectory

$$L_2=\int_0^T[\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t)+\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)+\mathbf{x}^T(t)\mathbf{S}(t)\mathbf{u}(t)$$
$$+\mathbf{q}^T(t)\mathbf{x}(t)+\mathbf{r}^T(t)\mathbf{u}(t)]dt \qquad (4)$$

without violating the linear system constraints:

$$\mathbf{E}_1(t)\mathbf{x}(t)+\mathbf{E}_2(t)\mathbf{u}(t)\le\mathbf{e}(t) \qquad (5)$$

It is assumed that the matrices **H**, **Q**, **R**, and **S** and the vectors **h**, **q**, and **r** are real and have appropriate dimensions with **H** and **Q** being positive-semidefinite and **S** being positive definite, **e** is an $O\times 1$ vector, $\mathbf{E}_1$ is an $O\times N$ matrix, and $\mathbf{E}_2$ is an $O\times J$ matrix. In addition, the terminal time T is assumed fixed.

Fourier-Based State Parameterization

The basic idea of Fourier-based state parameterization is to approximate each of the N state variables $x_n(t)$ by the sum of a third-order auxiliary polynomial $\delta_n(t)$ and a K term Fourier-type series, i.e., for $n=1,\ldots,N$,

$$x_n(t)=\delta_n(t)+\sum_{k=1}^{K}a_{nk}\cos\left(\frac{2k\pi t}{T}\right)+\sum_{k=1}^{K}b_{nk}\sin\left(\frac{2k\pi t}{T}\right) \qquad (6)$$

where

$$\delta_n(t)=\delta_{no}+\delta_{n1}t+\delta_{n2}t^2+\delta_{n3}t^3 \qquad (7)$$

The inclusion of the auxiliary polynomial in this representation ensures convergence on $[0, T]$ (not just $(0, T)$) for x_n and $\dot{x}_n$ and improves the speed of convergence (making it three orders faster) in comparison to a standard Fourier series expansion (Yen, 1989; Nagurka and Yen, 1990).

The four coefficients of the auxiliary polynomial $\delta_n(t)$ can be written as functions of state variable boundary values and Fourier coefficients.

$$\delta_{no}=x_{no}-\sum_{k=1}^{K}a_{nk},\ \delta_{n1}=\dot{x}_{no}-\frac{2\pi}{T}\sum_{k=1}^{K}kb_{nk} \qquad (8a,8b)$$

$$\delta_{n2}=3\left(x_{nT}-x_{no}+4\pi\sum_{k=1}^{K}kb_{nk}\right)T^{-2}-2(\dot{x}_{no}+\dot{x}_{nT})T^{-1} \qquad (8c)$$

$$\delta_{n3}=2\left(x_{nT}-x_{no}+2\pi\sum_{k=1}^{K}kb_{nk}\right)T^{-3}+(\dot{x}_{no}+\dot{x}_{nT})T^{-2} \qquad (8d)$$

where x_{no}, $\dot{x}_{no}$, x_{nT}, and $\dot{x}_{nT}$ are the boundary values

$$x_{no}=x_n(0),\ \dot{x}_{no}=\dot{x}_n(0),\ x_{nT}=x_n(T),\ \dot{x}_{nT}=\dot{x}_n(T) \qquad (9a\text{–}d)$$

Following substitution of equations (7) and (8) into (6), equation (6) can be rearranged and presented in the form

$$x_n(t)=\rho_1x_{no}+\rho_2\dot{x}_{no}+\rho_3x_{nT}+\rho_4\dot{x}_{nT}+\sum_{k=1}^{K}(\alpha_ka_{nk}+\beta_kb_{nk}) \qquad (10)$$

where

$$\rho_1=1-3\tau^2+2\tau^3,\ \rho_2=(\tau-2\tau^2+\tau^3)T \qquad (11a,b)$$

$$\rho_3=3\tau^2-2\tau^3,\ \rho_4=(-\tau^2+\tau^3)T \qquad (11c,d)$$

$$\alpha_k=\cos(2k\pi\tau)-1,\ \beta_k=\sin(2k\pi\tau)-2k\pi\tau(1-3\tau+2\tau^2) \qquad (11e,f)$$

with

$$\tau=\frac{t}{T} \qquad (12)$$

The parameters defined in equations (11a–f) are state independent and, since the terminal time T is known, are functions of time t only.

Equation (10) can be written compactly as

$$x_n(t)=\mathbf{c}^T(t)\mathbf{y}_n \qquad (13)$$

where

$$\mathbf{c}^T(t)=[\rho_1\ \rho_2\ \rho_3\ \rho_4\ \alpha_1\ \ldots\ \alpha_K\ \beta_1\ \ldots\ \beta_K] \qquad (14)$$

and

$$\mathbf{y}_n=[x_{no}\ \dot{x}_{no}\ x_{nT}\ \dot{x}_{nT}\ a_{n1}\ \ldots\ a_{nK}\ b_{n1}\ \ldots\ b_{nK}]^T$$
$$=[y_{n1}\ y_{n2}\ \ldots\ y_{nM}]^T \qquad (15)$$

are vectors of dimension $M=4+2K$. The first four elements of $\mathbf{y}_n$ are the values of x_n and $\dot{x}_n$ at the boundaries of [0, T]; the remaining elements are the coefficients of the Fourier-type series. Vector $\mathbf{y}_n$ can be viewed as a time-independent state parameter vector for $x_n(t)$ since it characterizes the trajectory of the n-th state variable over the time interval [0, T]. (Note that the first element is a given initial condition.) The ultimate objective is to determine the optimal state parameters for all state variables that minimize the performance index. This goal is achieved by first relating the state vector, its rate and the control vector to a state parameter vector for $\mathbf{x}(t)$.

The state vector containing the N state variables can be written as

$$\mathbf{x}(t)=\mathbf{C}(t)\mathbf{y} \tag{16}$$

where

$$\mathbf{C}=\begin{bmatrix}\mathbf{c}^T & & & \mathbf{0}\\ & \mathbf{c}^T & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{c}^T\end{bmatrix},\quad \mathbf{y}=\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \vdots\\ \mathbf{y}_N\end{bmatrix}=\begin{bmatrix}[y_{11}\ \dots\ y_{1M}]^T\\ [y_{21}\ \dots\ y_{2M}]^T\\ \vdots\\ [y_{N1}\ \dots\ y_{NM}]^T\end{bmatrix} \tag{17a,b}$$

Vector $\mathbf{y}$, the state parameter vector for $\mathbf{x}$, is a column vector of dimension NM; matrix $\mathbf{C}$ is a time dependent matrix of dimension $N\times NM$. By direct differentiation, the state rate vector can be written as

$$\dot{\mathbf{x}}(t)=\mathbf{D}(t)\mathbf{y} \tag{18}$$

where

$$\mathbf{D}(t)=\dot{\mathbf{C}}(t)=\begin{bmatrix}\mathbf{d}^T & & & \mathbf{0}\\ & \mathbf{d}^T & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{d}^T\end{bmatrix}, \tag{19}$$

The control vector $\mathbf{u}(t)$ can also be expressed as a function of $\mathbf{y}$. From equation (1)

$$\mathbf{u}(t)=\mathbf{B}^{-1}(t)\dot{\mathbf{x}}(t)+\mathbf{V}(t)\mathbf{x}(t) \tag{20}$$

where

$$\mathbf{V}(t)=-\mathbf{B}^{-1}(t)\mathbf{A}(t) \tag{21}$$

Substituting equations (16) and (18) into equation (20) gives

$$\mathbf{u}(t)=[\mathbf{B}^{-1}(t)\mathbf{D}(t)+\mathbf{V}(t)\mathbf{C}(t)]\mathbf{y} \tag{22}$$

Thus, using the Fourier-based state parameterization approach the state vector, state rate vector, and control vector can be represented as functions of the state parameter vector. It is shown in the following sections that by employing this representation LQ problems can be reformulated as QP problems with the elements of the state parameter vector $\mathbf{y}$ being the free variables.

Unconstrained LQ Problems

This section (i) demonstrates the conversion process from an unconstrained LQ problem to a QP problem via Fourier-based state parameterization, and (ii) develops an appropriate solution procedure. It is shown that the converted QP problem can be formulated as an unconstrained optimization problem with a quadratic objective function.

Conversion Process. The first step in the conversion is to rewrite the performance index as a function of state parameter vector $\mathbf{y}$. The terminal state part of the performance index L_1 can be written as a function of $\mathbf{y}$ by noting the following linear relation for the terminal state vector

$$\mathbf{x}(T)=\Theta\mathbf{y} \tag{23}$$

where Θ is a transformation matrix specified according to

$$\theta_{nm}=\begin{cases}1 & m=(n-1)M+3 \text{ for } n=1,\ \dots,\ N\\ 0 & \text{otherwise}\end{cases} \tag{24}$$

By substituting equation (23) into equation (3) the cost L_1 is

$$L_1=\mathbf{y}^T(\Theta^T\mathbf{H}\Theta)\mathbf{y}+\mathbf{h}^T\Theta\mathbf{y} \tag{25}$$

Similarly, the trajectory part of the performance index L_2 can be written as a function of $\mathbf{y}$, although the process is somewhat more complicated. Substituting equation (20) for the control vector into the integrand of equation (4) gives:

$$\mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}+\mathbf{x}^T\mathbf{S}\mathbf{u}+\mathbf{q}^T\mathbf{x}+\mathbf{r}^T\mathbf{u}=\mathbf{x}^T\mathbf{P}_1\mathbf{x}+\dot{\mathbf{x}}^T\mathbf{P}_2\dot{\mathbf{x}}+\dot{\mathbf{x}}^T\mathbf{P}_3\mathbf{x}+\mathbf{x}^T\mathbf{p}_1+\dot{\mathbf{x}}^T\mathbf{p}_2 \tag{26}$$

where

$$\mathbf{P}_1=\mathbf{Q}+\mathbf{V}^T\mathbf{R}\mathbf{V}+\mathbf{S}\mathbf{V},\ \mathbf{P}_2=\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1},\ \mathbf{P}_3=2\mathbf{B}^{-T}\mathbf{R}\mathbf{V}+\mathbf{B}^{-T}\mathbf{S} \qquad (27),(28),(29)$$

$$\mathbf{p}_1=\mathbf{q}+\mathbf{V}^T\mathbf{r},\ \mathbf{p}_2=\mathbf{B}^{-T}\mathbf{r} \qquad (30),(31)$$

where matrices $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ and vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ depend on system parameters and performance index weighting. Superscript $-T$ denotes inverse transpose. (For simplicity, the time-dependent symbol (t) has been omitted in the above equations.) By substituting equations (16) and (18) for the state vector and its rate, respectively, into equation (26), the integrand of the performance index can be expressed as a function of parameter vector $\mathbf{y}$, i.e.,

$$\mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}+\mathbf{x}^T\mathbf{S}\mathbf{u}+\mathbf{q}^T\mathbf{x}+\mathbf{r}^T\mathbf{u}=\mathbf{y}^T\Lambda\mathbf{y}+\mathbf{y}^T\Gamma \tag{32}$$

where

$$\Lambda=\mathbf{P}_1\otimes\mathbf{c}\mathbf{c}^T+\mathbf{P}_2\otimes\mathbf{d}\mathbf{d}^T+\mathbf{P}_3\otimes\mathbf{d}\mathbf{c}^T \tag{33}$$

$$\Gamma=\mathbf{p}_1\otimes\mathbf{c}+\mathbf{p}_2\otimes\mathbf{d} \tag{34}$$

In equations (33) and (34), $\otimes$ is a Kronecker product sign (see Brewer, 1978). Thus, from equation (32), the integral part of the performance index can be expressed as

$$L_2=\int_0^T(\mathbf{y}^T\Lambda\mathbf{y}+\Gamma^T\mathbf{y})dt=\mathbf{y}^T\Lambda^*\mathbf{y}+\Gamma^{*T}\mathbf{y} \tag{35}$$

where

$$\Lambda^*=\int_0^T\Lambda dt=\int_0^T(\mathbf{P}_1\otimes\mathbf{c}\mathbf{c}^T+\mathbf{P}_2\otimes\mathbf{d}\mathbf{d}^T+\mathbf{P}_3\otimes\mathbf{d}\mathbf{c}^T)dt \tag{36}$$

$$\Gamma^*=\int_0^T\Gamma dt=\int_0^T(\mathbf{p}_1\otimes\mathbf{c}+\mathbf{p}_2\otimes\mathbf{d})dt \tag{37}$$

For problems with time-varying system parameters and/or performance index weighting, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{p}_1$, and $\mathbf{p}_2$ are functions of time and the integrals of equations (36) and (37) can be evaluated numerically. For time-invariant problems, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{p}_1$, and $\mathbf{p}_2$ are constants and can be removed from the integrals, enabling the remaining integral parts of Λ^* and Γ^* to be evaluated analytically. That is, for time-invariant problems equations (36) and (37) can be rewritten as

$$\Lambda^*=\mathbf{P}_1\otimes\left[\int_0^T(\mathbf{c}\mathbf{c}^T)dt\right]+\mathbf{P}_2\otimes\left[\int_0^T(\mathbf{d}\mathbf{d}^T)dt\right]+\mathbf{P}_3\otimes\left[\int_0^T(\mathbf{d}\mathbf{c}^T)dt\right] \tag{38}$$

$$\Gamma^*=\mathbf{p}_1\otimes\left[\int_0^T\mathbf{c}dt\right]+\mathbf{p}_2\otimes\left[\int_0^T\mathbf{d}dt\right] \tag{39}$$

The integrals in the brackets of equations (38) and (39) have been evaluated in closed form (Yen, 1989). As a result, the Fourier-based approach is numerically integration-free in handling time-invariant problems.

In summary, by substituting equations (25) and (35) into equation (2), the performance index L can be written as a quadratic function of the state parameter vector $\mathbf{y}$.

$$L = \mathbf{y}^T \Omega^* \mathbf{y} + \mathbf{y}^T \omega^* \quad (40)$$

where

$$\Omega^* = \Theta^T \mathbf{H} \Theta + \Lambda^*, \; \omega^* = \Theta^T \mathbf{h} + \Gamma^* \quad (41),(42)$$

The optimization problem can thus be viewed as the search for the elements of $\mathbf{y}$, i.e., y_{nm}, $n = 1,\ldots, N$, $m = 1,\ldots, M$, that minimizes the performance index of equation (40) subject to the equality constraints

$$y_{n1} = x_{no} \text{ for } n = 1,\ldots, N \quad (43)$$

representing the initial conditions.

Solution Procedure. This subsection outlines an approach for solving the equality constrained QP problem (outlined above) by converting it into an unconstrained QP problem. To accomplish this goal, a new state parameter vector $\mathbf{z}$ is introduced, specified as

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \quad (44)$$

where

$$\mathbf{z}_1^T = [\mathbf{a}^T \mathbf{b}^T \dot{\mathbf{x}}_o^T \dot{\mathbf{x}}_T^T \mathbf{x}_T^T], \; \mathbf{z}_2 = \mathbf{x}_o \quad (45),(46)$$

with

$$\mathbf{x}_o = [x_{1o}\; x_{2o} \ldots x_{No}]^T, \; \dot{\mathbf{x}}_o = [\dot{x}_{1o}\; \dot{x}_{2o} \ldots \dot{x}_{No}]^T \quad (47),(48)$$

$$\mathbf{x}_T = [x_{1T}\; x_{2T} \ldots x_{NT}]^T, \; \dot{\mathbf{x}}_T = [\dot{x}_{1T}\; \dot{x}_{2T} \ldots \dot{x}_{NT}]^T \quad (49),(50)$$

$$\mathbf{a} = [a_{11} \ldots a_{1K}\; a_{21} \ldots a_{2K} \ldots\ldots a_{N1} \ldots a_{NK}]^T \quad (51)$$

$$\mathbf{b} = [b_{11} \ldots b_{1K}\; b_{21} \ldots b_{2K} \ldots\ldots b_{N1} \ldots b_{NK}]^T \quad (52)$$

Vector $\mathbf{z}_2$ contains the known initial values of the state vector; vector $\mathbf{z}_1$ is the remaining subset of the parameter vector $\mathbf{y}$ (i.e., obtained by excluding $\mathbf{z}_2$ from $\mathbf{y}$).

The two vectors $\mathbf{z}$ and $\mathbf{y}$ are related via a linear transformation

$$\mathbf{y} = \Phi \mathbf{z} \quad (53)$$

where Φ is a $NM \times NM$ matrix with elements 1 and 0. The performance index L of equation (40) can thus be rewritten as a function of $\mathbf{z}$

$$L = \mathbf{z}^T \Omega \mathbf{z} + \mathbf{z}^T \omega \quad (54)$$

where

$$\Omega = \Phi^T \Omega^* \Phi, \; \omega = \Phi^T \omega^* \quad (55),(56)$$

By expanding equation (54), the performance index can be expressed as

$$L = [\mathbf{z}_1^T \;\; \mathbf{z}_2^T] \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + [\mathbf{z}_1^T \;\; \mathbf{z}_2^T] \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \quad (57)$$

or equivalently

$$L = \mathbf{z}_1^T \Omega_{11} \mathbf{z}_1 + \mathbf{z}_1^T (\Omega_{12} + \Omega_{21}^T) \mathbf{z}_2 + \mathbf{z}_2^T \Omega_{22} \mathbf{z}_2 + \mathbf{z}_1^T \omega_1 + \mathbf{z}_2^T \omega_2 \quad (58)$$

The performance index of equation (58) is a quadratic function of $\mathbf{z}_1$, the unknown part of the state parameter vector. For an unconstrained LQ problem, the necessary condition of optimality can be obtained by differentiating the performance index with respect to this unknown state parameter vector. The result is the system of linear algebraic equations

$$(\Omega_{11} + \Omega_{11}^T) \mathbf{z}_1 = -(\Omega_{12} + \Omega_{21}^T) \mathbf{z}_2 - \omega_1 \quad (59)$$

from which the unknown vector $\mathbf{z}_1$ can be solved directly by a linear algebraic solver, such as a Gaussian elimination routine.

If the terminal condition of the state vector is known, the same solution procedure can be applied. The only modification is to redefine the unknown vector $\mathbf{z}_1$ as

$$\mathbf{z}_1 = [\mathbf{a}^T \; \mathbf{b}^T \; \dot{\mathbf{x}}_o^T \; \dot{\mathbf{x}}_T^T]^T \quad (60)$$

and the known vector $\mathbf{z}_2$ as

$$\mathbf{z}_2 = [\mathbf{x}_T^T \; \mathbf{x}_o^T]^T \quad (61)$$

Similarly, problems with fixed initial and/or terminal state rate vectors can handled.

An interesting feature of equation (59) is that the coefficient matrix of $\mathbf{z}_1$ is independent of known boundary values (i.e., $\mathbf{z}_2$). Thus, for the same unconstrained LQ problem with different boundary values, the coefficient matrix remains the same; only the right-hand side constant vector needs to be recomputed.

Linearly Constrained LQ Problems

In this section, the Fourier-based state parameterization approach is used to convert a linearly constrained LQ problem to a QP problem. In particular, the system constraints of equation (5) are converted into a system of linear algebraic constraints.

The approach is to substitute equation (20) into the inequality constraints of equation (5) giving

$$\mathbf{F}_1(t)\mathbf{x}(t) + \mathbf{F}_2(t)\dot{\mathbf{x}}(t) \le \mathbf{e}(t) \quad (62)$$

where

$$\mathbf{F}_1(t) = \mathbf{E}_1(t) + \mathbf{E}_2(t)\mathbf{V}(t), \; \mathbf{F}_2(t) = \mathbf{E}_2(t)\mathbf{B}^{-1}(t) \quad (63),(64)$$

Using the state parameterization of equations (16) and (18) in equation (62) gives

$$\mathbf{G}(t)\mathbf{y} \le \mathbf{e}(t) \quad (65)$$

where

$$\mathbf{G}(t) = \mathbf{F}_1(t)\mathbf{C}(t) + \mathbf{F}_2(t)\mathbf{D}(t) \quad (66)$$

The constraints of equation (65) are functions of time. Consequently, equation (65) represents an infinite number of constraints which need to be satisfied along the trajectory. For practicality, these constraints are relaxed to be satisfied only at a finite number of points (usually chosen to be equally spaced) in time. That is, equation (65) is replaced by a finite number of algebraic inequalities

$$\mathbf{G}(t_i)\mathbf{y} \le \mathbf{e}(t_i) \text{ for } i = 1,\ldots, I \quad (67)$$

where I is the number of sampling points for which the constraints need to be satisfied. In terms of the alternate state parameter vector $\mathbf{z}$, equation (67) can be rewritten using equation (53) as

$$\mathbf{G}^*(t_i)\mathbf{z} \le \mathbf{e}(t_i) \text{ for } i = 1,\ldots, I \quad (68)$$

where

$$\mathbf{G}^*(t_i) = \mathbf{G}(t_i)\Phi \quad (69)$$

By decoupling $\mathbf{z}$ into $\mathbf{z}_1$ and $\mathbf{z}_2$, the inequality constraints of equation (68) can be represented as

$$\begin{bmatrix} \mathbf{G}_{11}^*(t_i) & \mathbf{G}_{12}^*(t_i) \\ \mathbf{G}_{21}^*(t_i) & \mathbf{G}_{22}^*(t_i) \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \le \begin{bmatrix} \mathbf{e}_1(t_i) \\ \mathbf{e}_2(t_i) \end{bmatrix} \text{ for } i = 1,\ldots, I \quad (70)$$

Since $\mathbf{z}_2$ is known, the corresponding terms can be moved to the right-hand side of equation (70) giving

$$\begin{bmatrix} \mathbf{G}_{11}^*(t_i) \\ \mathbf{G}_{21}^*(t_i) \end{bmatrix} \mathbf{z}_1 \le \begin{bmatrix} \mathbf{e}_1(t_i) - \mathbf{G}_{12}^*(t_i)\mathbf{z}_2 \\ \mathbf{e}_2(t_i) - \mathbf{G}_{22}^*(t_i)\mathbf{z}_2 \end{bmatrix} \text{ for } i = 1,\ldots, I \quad (71)$$

Thus, the system constraints of equation (5) can be approximated by the linear algebraic inequalities of equation (71).

In summary, by applying Fourier-based state parameterization, a linearly constrained LQ problem can be converted into a QP problem in which the quadratic function of equation (58) is to be minimized without violating the system of linear algebraic constraints of equations (43) and (71).

Fourier-Based Approach for General Linear Systems

The approach presented above is applicable to systems with square and invertible control influence matrices. For general linear systems, the control influence matrix $\mathbf{B}$, is an $N \times J$

matrix where the number of state variables, N, is greater than the number of control variables, J. This section generalizes the Fourier-based approach to the more common case of general linear systems which have fewer control variables than state variables. It is assumed that the rank of $\mathbf{B}$ is equal to J.

To apply the Fourier-based approach, the state-space model of equation (1) is first modified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}'(t)\mathbf{u}'(t) \tag{72}$$

where

$$\mathbf{B}'(t) = \mathbf{B}'_{N\times N} = \begin{bmatrix} \mathbf{I}_{(N-J)\times(N-J)} & \\ & \mathbf{B}_{N\times J} \\ \mathbf{O}_{J\times(N-J)} & \end{bmatrix} \tag{73}$$

and

$$\mathbf{u}'(t) = \mathbf{u}'_{N\times 1} = \begin{bmatrix} \hat{\mathbf{u}}_{(N-J)\times 1} \\ \mathbf{u}_{J\times 1} \end{bmatrix} \tag{74}$$

with the subscripts representing the dimensions of the matrices. By introducing an artificial control vector, $\hat{\mathbf{u}}(t)$, the new control influence matrix, $\mathbf{B}'$, can be inverted enabling the calculation of the control, $\mathbf{u}'(t)$, for any given trajectory (similar to equation (20)). Note that it can be guaranteed that $\mathbf{B}'$ is invertible if the last J rows of $\mathbf{B}$ are nonsingular. However, if the last J rows are singular, the first $(N-J)$ columns of $\mathbf{B}$ in equation (73) can always be modified to make it invertible since it has been assumed that $\mathbf{B}$ has rank J.

In order to predict the optimal solution, the performance index (for the case $\mathbf{S}=\mathbf{0}$, $\mathbf{q}=\mathbf{0}$) is modified to

$$L' = L + w\int_0^T [\hat{\mathbf{u}}^T(t)\hat{\mathbf{u}}(t)]dt \tag{75}$$

where L is the performance index of the original LQ problem and w is a weighting constant chosen to be a large positive number. The integral term associated with w is used to represent the contribution of the artificial control.

The advantage of using artificial control variables is that a nonactively controlled system can be converted into an actively controlled system to which the Fourier-based state parameterization approach is applicable. The tradeoff is that the resulting solution will not, in a strict mathematical sense, satisfy the trajectory admissibility requirement (see Yen and Nagurka, 1988) due to the existence of the physically non-existent artificial control. However, by employing the penalty function of equation (75), the magnitude and influence of the artificial control variables can be made insignificant and the solution of the modified optimal control problem can closely approximate the solution of the original LQ problem.

Simulation Studies

In the simulation studies reported here, the solutions of time-invariant LQ problems are obtained by Fourier-based state parameterization and compared with closed-form optimal solutions or solutions from standard numerical algorithms. Example 1 is designed to study the effectiveness of the Fourier-based approach in solving unconstrained LQ problems. Examples 2 and 3 are used to study the effectiveness of Fourier-based state parameterization in handling linearly constrained LQ problems. In particular, Example 2 considers a LQ problem with a linear state constraint, whereas Example 3 examines a problem with a bounded control variable.

In the first example, the Fourier-based approach is compared to a transition matrix approach, which was applied to generate the state and control variables at prespecified equally-spaced points in time. An overview of the transition matrix approach for unconstrained LQ problems is presented in Appendix A; additional details can be found in (Speyer, 1986). In the last two examples, the QP solution algorithm developed by Gill and Murray (1977), considered to be one of the most efficient algorithms for QP problems, was implemented to determine the optimal state parameters of the Fourier-based approach.

Efforts were made to optimize the speed of the computer codes, all of which were written in "C." The simulations were executed on a SUN-3/60 workstation.

Example 1. This example considers an N input Nth order linear time-invariant dynamic system expressed in canonical form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{x}^T(0) = [1\ 2\ \dots\ N] \tag{76}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 1 & -2 & \dots & & (-1)^{N+1}N \end{bmatrix},\ \mathbf{B} = \mathbf{I}_{N\times N} \tag{77}$$

The problem is to determine the control vector $\mathbf{u}$ that minimizes the performance index

$$L = \mathbf{x}^T(1)\mathbf{H}\mathbf{x}(1) + \int_0^1 (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt,$$
$$\mathbf{H} = 10\mathbf{I}_{N\times N},\ \mathbf{Q} = \mathbf{R} = \mathbf{I}_{N\times N} \tag{78}$$

A computationally efficient method for solving this unconstrained LQ problem is the transition matrix approach described in Appendix A. The transition matrix approach converts an optimal control problem into a linear TPBVP (such as equation (A-9)). By evaluating the transition matrix of this boundary value problem, the problem can be reformulated as an initial value problem. In this study, the transition matrices were computed numerically using the algorithm presented in (Franklin and Powell, 1980, pp. 176–177). The system response was obtained at 50 equally-spaced points.

This unconstrained LQ problem could also be solved by integrating the Riccati equation. Although the Riccati method puts the optimal solution in closed-loop form and is thus a preferred approach for physical implementation, it is computationally more intensive than the (open-loop) transition matrix approach. Since the design of an optimal LQ controller is often an iterative process, the transition matrix approach is thus a more efficient investigative tool than a Riccati equation solver.

In addition to the transition matrix approach, the Fourier-based approach involving a two-term Fourier-type series was used to solve this problem. The integrals of equations (38) and (39) were determined directly via table look-up. The linear algebraic equations (59) representing the condition of optimality were solved using a Gauss-Jordan elimination routine for the optimal state parameter vector. This vector was used in equation (54) to determine the value of the performance index.

Simulation results for $N=2, 4, \dots, 20$ are summarized in Table 1 where execution time (in seconds) is used as an index of computational efficiency. The results demonstrate that the Fourier-based approach is both efficient, especially in solving for the optimal control of high order systems, and accurate (i.e., the error of the performance index is always less than 1 percent). In comparison to the transition matrix approach, the Fourier-based method is increasingly more efficient for $N>6$. For $N=20$ the Fourier-based results suggest a 39 percent reduction in execution time. For $N\leq 6$, the Fourier-based method is less efficient than the transition matrix approach, since the time to evaluate the integrals from the table look-up, a fixed time for any order system, is a significant fraction of the overall computational cost. For high order systems the principal computational cost is due to the solution of the linear algebraic

Table 1 Summary of simulation results of example 1

	Transition-Matrix Approach		Fourier-Based Approach[a]		Comparison	
N	Performance Index	Time	Performance Index	Time	%Time[b]	$\%\Delta L^c$
2	5.3591	0.18	5.3591	0.30	167	$< 3.7 \times 10^{-5}$
4	44.250	0.66	44.250	0.86	130	$< 1.1 \times 10^{-3}$
6	153.76	1.88	153.76	1.96	104	$< 3.9 \times 10^{-3}$
8	373.02	4.26	373.06	3.80	89	$< 1.1 \times 10^{-2}$
10	741.61	7.94	741.77	6.46	81	$< 2.2 \times 10^{-2}$
12	1299.4	14.14	1299.9	10.67	75	$< 3.6 \times 10^{-2}$
14	2086.4	22.61	2087.7	16.04	71	$< 6.4 \times 10^{-2}$
16	3142.8	34.34	3145.8	22.84	67	$< 9.5 \times 10^{-2}$
18	4509.1	48.40	4514.9	31.28	64	$< 1.3 \times 10^{-1}$
20	6225.4	68.95	6235.9	42.20	61	$< 1.7 \times 10^{-1}$

[a]With single segment two-term Fourier-type series
[b]Percent of execution time of Fourier-based approach relative to execution time of transition-matrix approach
[c]Percent difference of performance index of Fourier-based approach relative to performance index value of transition-matrix approach

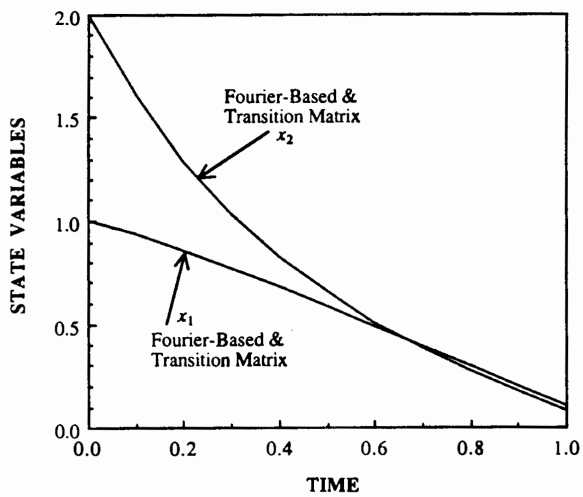

Fig. 1(a) State variable histories for example 1

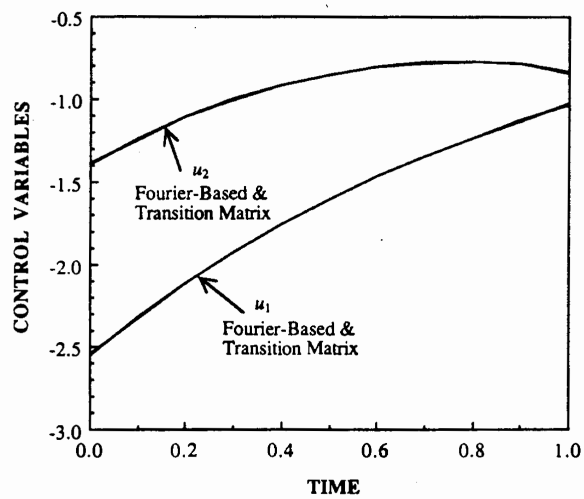

Fig. 1(b) Control variable histories for example 1

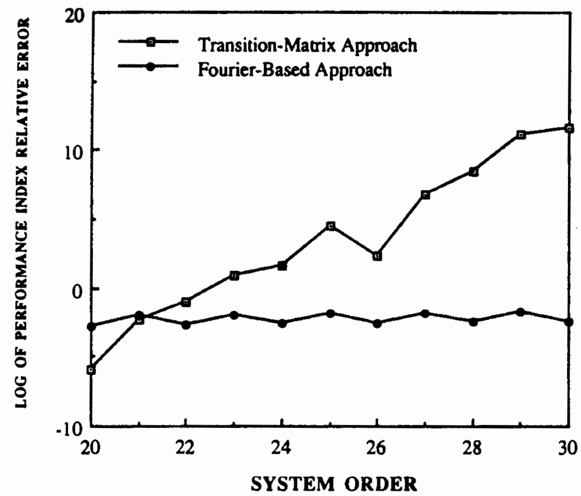

Fig. 2 Logarithm of relative error of performance index versus system order for example 1

equations (59), which is less intensive than the solution via the transition matrix method.

The time histories of the state and control variables for the case $N=2$ are plotted in Figs. 1(*a*) and 1(*b*), respectively. The response curves from the transition matrix and Fourier-based approaches drawn in these figures are almost indistinguishable. Hence, the Fourier-based solution achieves convergence on the trajectories of the state and control variables as well as on the performance index.

In addition to improved computational efficiency with satisfactory accuracy, the Fourier-based approach is robust in generating the system response for high order systems. To study robustness, simulations were conducted for the high-order cases $N=20, 21, \ldots, 30$ using the transition matrix and Fourier-based approaches. The results shown in Figure 2 suggest that the transition matrix approach becomes unstable for such high order systems. The figure shows a measure of the relative error of the performance index ($\log_{10}(\Delta L/L)$ where ΔL is the absolute difference of the values of the performance index from the Fourier-based and transition matrix solutions) as a function of N. In the Fourier-based approach the relative error is roughly constant as the system order becomes high. In contrast, in the transition matrix approach, the relative error increases (trendwise) with system order and becomes quite significant. For example, for $N \geq 23$, the relative error measure $\log_{10}(\Delta L/L)$ is larger than zero, implying that the relative error of the performance index predicted by the transition matrix approach exceeds 100 percent! The error increases dramatically with system order, indicating that a numerical instability problem has been encountered. This problem is caused principally by the error in computing the state transition matrix of the TPBVP (equation (A-9)). There does not seem to be a computationally efficient solution approach to overcome this numerical difficulty (Moler and Loan, 1978).

In summary, the Fourier-based approach offers advantages in terms of computational efficiency and numerical robustness relative to the transition matrix approach.

Example 2. This example, adapted from (Evtushenko, 1985, p. 438), considers a time-invariant LQ problem with a time-varying state constraint. The system involving two state variables and a single control variable is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad (79)$$

It is required to find the solution that minimizes the performance index

$$L = \int_0^1 [x_1^2 + x_2^2 + 0.005u^2]dt \quad (80)$$

without violating the constraint

$$x_2(t) \leq e(t) \quad (81)$$

where

$$e(t) = 8(t-0.5)^2 - 0.5 \quad (82)$$

In (Evtushenko, 1985), this problem was solved using a

Table 2 Summary of simulation results of example 2

K	Performance Index
3	0.17480
4	0.17268
5	0.17115
6	0.17069
7	0.17069
8	0.17028
9	0.17013

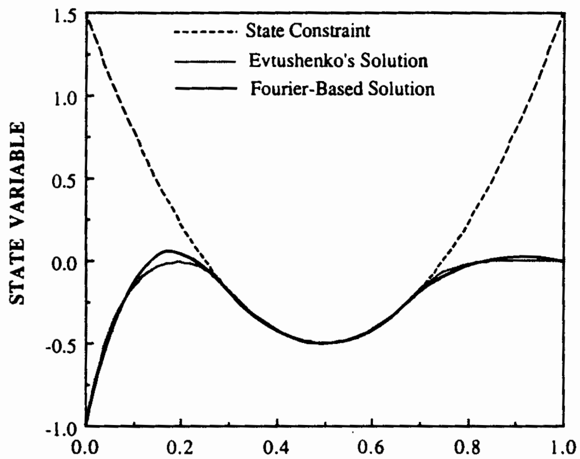

Fig. 3 State variable x_2 history for example 2

control parameterization approach. Here, the problem was solved using the Fourier-based approach for a general linear system with a weighting coefficient of $w = 10^5$ to penalize the artificial control. The resulting values of the performance index for three to nine term Fourier-type series are summarized in Table 2. As shown in this Table, the performance index values decrease as the number of terms of the Fourier-type series increases. In particular, the Fourier-based solutions with series of six and more terms are less than the minimum performance index of 0.17114 obtained by Evtushenko (1985). Furthermore, the differences between the Fourier-based solutions are small (for example, the difference between the eight and nine term Fourier-based solutions is less than (0.09 percent) suggesting that convergence has been achieved.

The response history for $x_2(t)$ obtained with a three term Fourier-type series is plotted in Fig. 3. The constraint history and the solution computed by Evtushenko (1985) are also plotted in this figure. The Fourier-based solution satisfies the state constraint and closely approximates the trajectory predicted by Evtushenko. In fact, the Fourier-based solution appears indistinguishable from Evtushenko's solution when the state constraint is active. To verify that the artificial control variable technique is successful, the history of the artificial control variable for a three term Fourier-based solution has been plotted in Fig. 4. As shown in the figure, the artificial control variable is small in magnitude (on the order of 10^{-5}). Hence, it has minimal influence on system performance.

In summary, this example demonstrates the applicability of the Fourier-based approach for handling LQ problems with state inequality constraints. For the problem studied, the Fourier-based approach yields higher accuracy in predicting the optimal solution in comparison to a previous result.

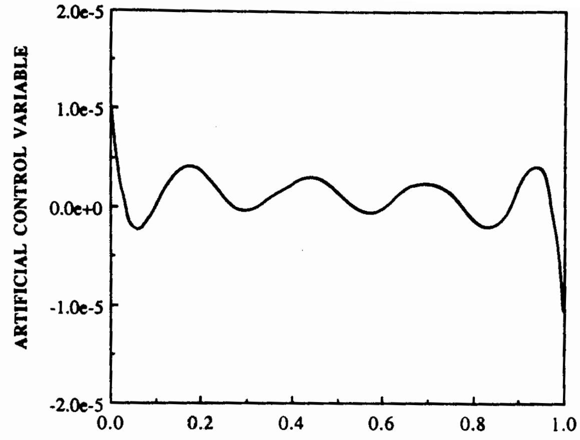

Fig. 4 Artificial control history for example 2

Table 3 Summary of simulation results of example 3

K	Performance Index
3	8.309
4	7.997
5	7.014
6	6.611
7	6.477
8	6.307
9	6.152

Example 3. This example, adapted from (Leondes and Wu, 1971), considers another single input second order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u , \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.231 \\ 1.126 \end{bmatrix} \tag{83}$$

The performance index is

$$L = \frac{1}{2} \int_0^5 [x_1^2 + x_2^2] dt \tag{84}$$

A constraint is imposed on the control variable

$$|u| \le 0.8$$

The optimal solution, as computed by Leondes and Wu (1971), has a bang-bang nature, i.e.,

$$u(t) = \begin{cases} -0.8 \text{ for } 0 \le t < 1.275 \\ 0.8 \text{ for } 1.275 < t \le 5.0 \end{cases} \tag{86}$$

The corresponding value of the performance index is 5.660.

This problem was first solved using the Fourier-based approach for a general linear system with a penalty weighting coefficient of $w = 10^5$. The values of the performance index obtained by using a three to nine term Fourier-type series are tabulated in Table 3. The near optimal solutions generated by the Fourier-based approach converge to the optimal bang-bang solution as the number of terms of the Fourier-type series increases. However, the speed of convergence is quite slow (with a three term Fourier-series the error in the performance index is 47 percent; with a nine term series the error is 9 percent).

This phenomenon of slow convergence is also evident in the control variable histories. The histories obtained using three, six, and nine term Fourier-series are plotted in Fig. 5(*a*). The bang-bang optimal solution is also provided in this figure. The

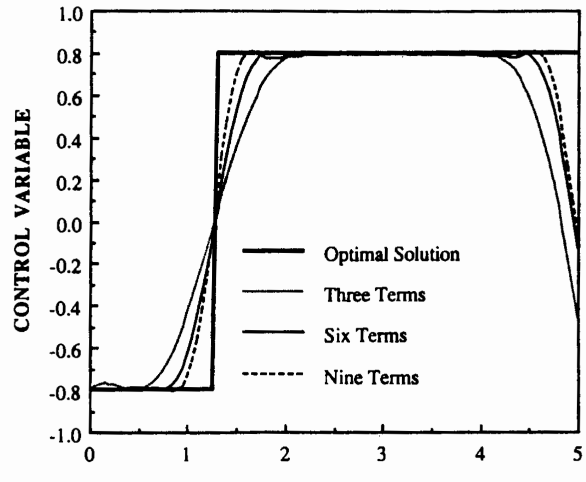

Fig. 5(a) Control variable histories for example 3

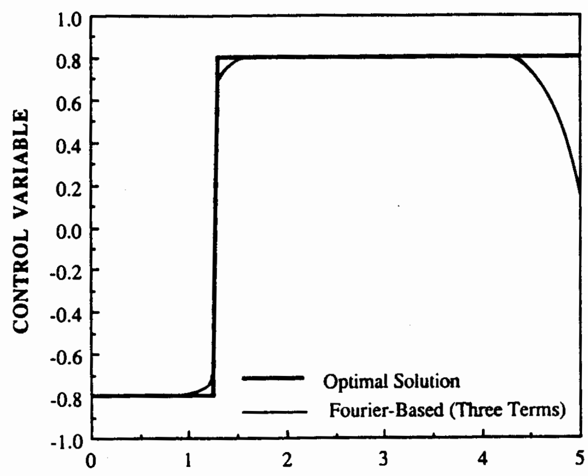

Fig. 5(b) Control variable histories for example 3 (two segment)

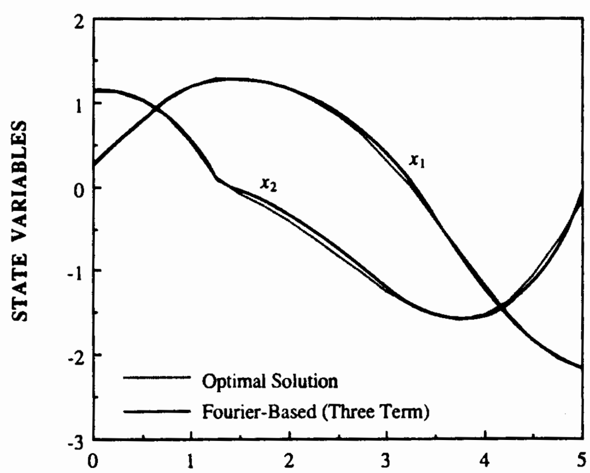

Fig. 5(c) State variable histories for example 3 (two segment)

principal reason for the slow convergence of the Fourier-based approach is due the instantaneous switch of the optimal control solution at $t = 1.275$. The Fourier-based solution assumes continuity of each state variable, each state variable rate, and hence each control variable throughout the trajectory. Since this assumption is violated, significant discrepancies between the optimal and Fourier-based solutions can be observed in the neighborhood of the finite jump.

One remedy of this slow convergence is to generalize the "single" segment Fourier-based approach developed in this paper to a "multiple" segment Fourier-based approach which employs spline-like Fourier-based approximations over the trajectory. The idea is to first estimate the location of the instantaneous jumps by using the single segment Fourier-based approach, and then approximate each continuous part of the trajectory by a unique Fourier-based representation. For instance, from the results of Fig. 5(*a*), the time interval [0, 5] is divided into [0, 1.3] and [1.3, 5.0], each of which is then represented by a unique three-term Fourier-based approximation. Additionally, equality constraints are introduced to ensure the continuity of the state variable response between the two segments. The resulting control variable response is plotted in Fig. 5(*b*).

The multiple segment Fourier-based solution of Fig. 5(*b*) simulates the finite jump with greater accuracy than the single segment solution of Fig. 5(*a*). The performance index of the multiple segment solution is 5.832 which has a 3 percent error compared to Leondes and Wu's optimal value (and is less than the single segment solutions listed in Table 3). Judging from the value of the performance index, the quality of the Fourier-based solution is most sensitive to the changes at the finite jump and least sensitive to the deviation from the optimal solution $4 < t < 5$. Since the performance index is a function only of the state variables, this claim can be verified by examining the response of the state variables, shown in Fig. 5(*c*). The state variable trajectories of the optimal and Fourier-based solutions are in close agreement. Thus, despite the seemingly poor prediction of the control variable for $4 < t < 5$, the state variable histories as well as the value of the performance index are determined satisfactorily. In summary, by employing a multiple segment Fourier-based approach, accurate near optimal solutions of bang-bang control problems can be obtained.

Due to its mathematical complexities, the methodology of the multiple segment Fourier-based approach is not developed in this paper. For applications of the multiple segment approach to unconstrained LQ problems, readers are referred to (Yen, 1989; Yen and Nagurka, 1989).

Discussion

An advantage of a state parameterization approach, such as the Fourier-based approach, is that it characterizes the optimal trajectory (which, in theory, consists of an infinite number of points) by a relatively small number of state parameters. An optimal control problem can thus be converted into an algebraic optimization problem (i.e., a MP problem). In general, the corresponding computations are much less complicated than those involved in standard optimal control solvers.

For unconstrained LQ problems, the performance index, which initially is written as a quadratic functional, is converted into a quadratic function. By differentiating this quadratic function with respect to the free parameters, the necessary condition of optimality is derived as a system of linear algebraic equations which can readily be solved. As verified by simulation results, this Fourier-based approach is computationally more efficient than a standard LQ problem solver (based on the transition matrix approach) in handling time-invariant LQ problems.

For linearly constrained LQ problems, the system constraints are relaxed to be satisfied only at a finite number of points (usually equally-spaced) in time. Consequently, the linear system constraints are replaced by a finite number of linear algebraic inequalities. The optimal control problem is thus converted into a QP problem.

In applying the Fourier-based approach, finite-term Fourier-type series are employed. As a result, the Fourier-based approach can be classified as a near-optimal (or suboptimal) control approach. The accuracy of the Fourier-based approach can be estimated empirically by increasing the number of terms of the Fourier-type series. Additional terms can be added, on a term by term basis, until the value of the performance index converges, indicating that the optimal solution is closely approximated. For problems with bang-bang characteristics (e.g., Example 3), discontinuities of the optimal solution often cause slow convergence of the Fourier-based solution. As suggested in Example 3, this problem can be overcome by replacing the "single" segment approximation by a "multiple" segment approximation.

Work in progress is generalizing the Fourier-based approach for solving general nonlinear optimal control problems (Yen, 1989). The underlying idea is similar to sequential quadratic programming, a method which converts a nonlinear programming problem into a sequence of QP problems. Similarly, a nonlinear optimal control problem can be converted into a sequence of linearly constrained LQ problems each of which can be solved by an efficient and robust solver, such as the proposed approach.

Conclusion

Based on the idea of state trajectory parameterization, this paper develops a Fourier-based approach for solving unconstrained and linearly constrained LQ optimal control problems. It is shown that LQ problems can be converted into QP problems. In particular, the necessary condition of optimality for unconstrained LQ problems is obtained as a system of linear algebraic equations.

Simulation results indicate that the Fourier-based approach is computationally more efficient and numerically more robust than the transition matrix approach in handling high order unconstrained LQ problems. The results also show that, in many cases, the Fourier-based approach provides sufficient accuracy for many linearly constrained LQ problems. However, slow convergence has been observed in applying the Fourier-based approach to problems with discontinuous optimal solutions. This difficulty has been overcome by generalizing the approach from a single segment approximation to a multiple segment approximation. In summary, the Fourier-based state parameterization approach promises to be an effective and general computational tool for solving linearly constrained LQ problems.

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APPENDIX A

Transition Matrix Approach for Unconstrained LQ Problems

Consider the LQ problem that minimizes

$$L=\frac{1}{2}\,\mathbf{x}^T(T)\mathbf{H}\mathbf{x}(T)+\frac{1}{2}\int_0^T(\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t)+\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t))dt \qquad \text{(A-1)}$$

subject to

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{B}(t)\mathbf{u}(t),\ \mathbf{x}(0)=\mathbf{x}_o \qquad \text{(A-2)}$$

For simplicity, cross product and linear terms of the control and state vectors have been omitted from the performance index. The order of the system is assumed to be N.

In the transition matrix approach (see, for example, Speyer, 1986), the Hamiltonian is first introduced as

$$H=\frac{1}{2}\,\mathbf{x}^T\mathbf{Q}\mathbf{x}+\frac{1}{2}\,\mathbf{u}^T\mathbf{R}\mathbf{u}+\lambda^T\mathbf{A}\mathbf{x}+\lambda^T\mathbf{B}\mathbf{u} \qquad \text{(A-3)}$$

where λ can be viewed as a Lagrange multiplier vector whose elements are often called costate variables. It can be shown that the necessary conditions of optimality are

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} = \mathbf{Ax} + \mathbf{Bu},\ \mathbf{x}(0) = \mathbf{x}_o \qquad \text{(A-4)}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} = -\mathbf{Qx} - \mathbf{A}^T\boldsymbol{\lambda},\ \boldsymbol{\lambda}(T) = \mathbf{Hx}(T) \qquad \text{(A-5)}$$

$$0 = \frac{\partial H}{\partial \mathbf{u}} = \mathbf{Ru} + \mathbf{B}^T\boldsymbol{\lambda} \qquad \text{(A-6)}$$

Equation (A-6) can be solved for the optimal control **u** giving

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda} \qquad \text{(A-7)}$$

Substituting equation (A-7) into equation (A-4) yields

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{BR}^{-1}\mathbf{B}^T\boldsymbol{\lambda} \qquad \text{(A-8)}$$

Combining equations (A-5) and (A-8) gives a TPBVP that consists of $2N$ linear homogeneous differential equations

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BR}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix},\ \mathbf{x}(0) = \mathbf{x}_o,\ \boldsymbol{\lambda}(T) = \mathbf{Hx}(T) \qquad \text{(A-9)}$$

This system of equations is often called the Hamiltonian system. Its solution has the following form

$$\begin{bmatrix} \mathbf{x}(t_2) \\ \boldsymbol{\lambda}(t_2) \end{bmatrix} = \phi(t_2, t_1) \begin{bmatrix} \mathbf{x}(t_1) \\ \boldsymbol{\lambda}(t_1) \end{bmatrix} \qquad \text{(A-10)}$$

where ϕ is the transition matrix of the Hamiltonian system. By setting $t_2 = T$ and $t_1 = 0$, equation (A-10) gives

$$\begin{bmatrix} \mathbf{x}(T) \\ \boldsymbol{\lambda}(T) \end{bmatrix} = \begin{bmatrix} \phi_{11}(T,0) & \phi_{12}(T,0) \\ \phi_{21}(T,0) & \phi_{22}(T,0) \end{bmatrix} \begin{bmatrix} \mathbf{x}(0) \\ \boldsymbol{\lambda}(0) \end{bmatrix} \qquad \text{(A-11)}$$

With the terminal condition $\boldsymbol{\lambda}(T) = \mathbf{Hx}(T)$ given by equation (A-5), $\boldsymbol{\lambda}(0)$ can be determined from equation (A-11) as

$$\boldsymbol{\lambda}(0) = \mathbf{K}(T)\mathbf{x}(0) \qquad \text{(A-12)}$$

where

$$\mathbf{K}(T) = [\phi_{22}(T,0) - \mathbf{H}\phi_{12}(T,0)]^{-1}[\mathbf{H}\phi_{11}(T,0) - \phi_{21}(T,0)] \qquad \text{(A13)}$$

The Hamiltonian system of equation (A-9) can thus be viewed as an initial value problem. Using equation (A-10), the solution of this initial value problem can be formulated as

$$\begin{bmatrix} \mathbf{x}(t_p + \Delta t) \\ \boldsymbol{\lambda}(t_p + \Delta t) \end{bmatrix} = \phi(t_p + \Delta t,\ t_p) \begin{bmatrix} \mathbf{x}(t_p) \\ \boldsymbol{\lambda}(t_p) \end{bmatrix} \text{ for } p = 1, \ldots, P \qquad \text{(A-14)}$$

where P is the number of equally-spaced points for which the solution is required and $\Delta t = T/P$. Note that for time-invariant problems, the transition matrix $\phi(t_p + \Delta t,\ t_p)$ is independent of t_p and is only a function of Δt. A solution approach based on equation (A-14) is computationally much more efficient in general than solving equation (A-11) using numerical integration-based differential equation solvers such as Runge-Kutta methods. The corresponding optimal control **u** can be computed from equation (A-7). Using this transition-matrix approach, it can also be shown that the corresponding performance index value is

$$L_{\text{optimal}} = \frac{1}{2}\mathbf{x}^T(0)\mathbf{K}(T)\mathbf{x}(0) \qquad \text{(A-15)}$$