Linear Quadratic Optimal Control
Via Fourier-Based State Parameterization

A method for determining the optimal control of unconstrained and linearly con-
strained linear dynamic systems with quadratic performance indices is presented.
The method is based on a modifled Fouier series approximation of each state vari-
able that converts the linear quadratic (LQ) problem into a mathematical program-
ing problem. In particular, it is shown that an unconstrained LQ problem can be cast
as an unconstrained quadratic programming problem where the necessary condi-
tion of optimality is derived as a system of linear algebraic equations. Furthermore,
it is shown that a linearly constrained LQ problem can be converted into a general
quadratic programming problem. Simulation studies for constrained LQ systems,
including a bang-bang control problem, demonstrate that the approach is accurate.
The results also indicate that in solving high-order unconstrained LQ problems the
approach is computationally more efficient and robust than standard methods.

Introduction
The optimal control of linear, lumped parameter, dynamic systems is the subject of
much theoretical and practical interest, and is well-covered in many textbooks (e.g.,
Athans and Falb, 1966; Kirk, 1970; Sage and White, 1977; Lewis, 1986). Typi-
ically, the necessary condition of optimality is formulated as a two-point boundary-
value problem (TPBVP) using variational methods. Except in some special cases, the
solution of this TPBVP is usually difficult, and in some cases not practical, to
obtain.

In contrast to variational methods, trajectory parameteri-
ization represents a distinct approach toward the solution of
optimal control problems. In general, these techniques ap-
proximate the control and/or state vectors by functions with
unknown coefficients, thereby converting an optimal control
problem into a mathematical programming (MP) problem. A
near optimal solution can then be obtained via various well
developed optimization algorithms. For example, quadratic
programming has been used to solve parameterized linear op-
timal control problems (Canon and Eaton, 1966; Blum and
Pigford, 1968; Ijima and Sakakibara, 1969; Borrage and Johnson, 1970;
can be found in (Kraft, 1960). Theoretical aspects of solving optimal control
problems via trajectory parameterization are also covered in
(Canon et al., 1970; Tahah and Kuo, 1971; Lundberg, 1972;
Evenshen, 1965).

A direct application of trajectory parameterization is to par-
meterize the control variables. For example, after representing

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Finally, combined state and control parameterization approaches have been suggested. In (Vlassenbroeck and Van Dooren, 1988), both the state and control variables are expanded in Chebyshev series. Although their approach can handle linear as well as nonlinear problems, it requires the system dynamics, boundary conditions, and performance index, involves tedious analytical formula for different optimal control problems, and increases the number of unknown variables of the converted M-P problem. In general, the number of free variables is typically higher than the number employed in either state or control parameterization approaches.

Of the different trajectory parameterization approaches, state parameterization offers two major advantages. First, boundary condition requirements on the state variables can be handled directly. Second, if the trajectory inadmissibility problem can be overcome, the state equations can be used as algebraic equations. As a result, the process of determining the control functions in implicit relationships between the state and control vectors is easier to implement in state parameterization than in control parameterization.

This research is part of a broader effort toward the development of a computational tool for solving optimal control problems via state parameterization. As part of this effort, this paper presents a specialized version of the Fourier-based state parameterization approach (Nagurka and Yet, 1990) for determining the optimal trajectories of linear systems described by state-space models with quadratic performance indices and linear constraints. The approach employs a third-order polynomial approximated to a Fourier-type series to represent each state variable. For the unconstrained optimal control problem, a system of linear algebraic equations is derived as the condition of optimality from which the near optimal state and control trajectories can be determined. For the linearly constrained problem, the LQ problem is converted to a quadratic programming problem which can be solved by well developed routines. Example problems demonstrate the high accuracy, computational efficiency, and robustness of the method. Simulation studies for constrained LQ systems are included in two examples, one of which addresses the applicability of a multiple model (i.e., updated) Fourier-based method for solving bang-bang control problems.

Problem Statement

The behavior of a linear dynamic system is described by the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x(t)$ is the state vector, $u(t)$ is the control variable, and $A$ and $B$ are real constant matrices. For simplicity, the system is assumed to be controllable and of full rank.

$$L^*(t) = x(t)$$

where $L^*$ is the cost associated with the optimal terminal state $L^*(T) = E_x[T]$. The control space $u(t)$ is considered to be active.

$$L^*(t) = x(t)$$

and $L^*$ is the cost associated with the trajectory

$$L^* = \frac{1}{2} \int_0^T \left[ \dot{x}(t)^T Q(x(t)) + u(t)^T R u(t) \right] dt$$

without violating the linear system constraints:

$$E_x(t^T) = E_x(t^T) = x(t^T)$$

where $Q$ and $R$ are positive semidefinite and $u(t)$ is positive definite, $e$ is an $N \times 1$ vector, $E_x$ is an $N \times N$ matrix, and $E_x(t^T)$ is an $N \times T$ matrix. In addition, the terminal time $T$ is assumed fixed.

Fourier-Based State Parameterization

The basic idea of Fourier-based state parameterization is to approximate each of the $N$ state variables $x(t)$ by the sum of a third-order auxiliary polynomial $\phi(t)$ and a $K$ term Fourier-type series, i.e., for $n = 1, \ldots, N$

$$x(t) = \phi(t) + \sum_{k=1}^{K} \alpha_{k} \cos \left( \frac{2 \pi k t}{T} \right)$$

where

$$\phi(t) = \sum_{k=1}^{K} \alpha_{k} \cos \left( \frac{2 \pi k t}{T} \right)$$

The inclusion of the auxiliary polynomial in this representation ensures convergence on $[0, T]$ (not just on $(0, T)$) for $x$ and $\dot{x}$, and improves the speed of convergence (making it three orders faster) in comparison to a standard Fourier series expansion (Yen, 1989; Nagurka and Yet, 1990).

The four coefficients of the auxiliary polynomial $\phi(t)$ can be written in terms of state vector values including Fourier coefficients.

$$\phi(t) = \sum_{n=0}^{N-1} \phi(t)$$

where $\phi(t)$ are the boundary values $x(t_0)$, $x(t_0 + T)$, $x(t_0 + 2T)$, $x(t_0 + 3T)$, $x(t_0 + 4T)$, $x(t_0 + 5T)$, and $x(t_0 + 6T)$.

The following substitution of equations $(7)$ and $(8)$ into $(6)$, equation $(6)$ is rearranged and presented in the form

$$x(t) = x(t_0) + \sum_{k=1}^{K} \alpha_{k} \cos \left( \frac{2 \pi k t}{T} \right)$$

where $\alpha_{k}$ are the Fourier coefficients $\phi(t)$.

$$\gamma_{k} = \sum_{n=0}^{N-1} \phi(t)$$

The parameters defined in equations $(14)$ and $(15)$ are state independent and, since the terminal time $T$ is known, functions of time only.

$$\phi(t) = x(t_0) + \sum_{k=1}^{K} \alpha_{k} \cos \left( \frac{2 \pi k t}{T} \right)$$

Equation $(10)$ can be written compactly as

$$\phi(t) = \Gamma(t) \gamma$$

where

$$\phi(t) = \gamma_{k}$$

and $\gamma_{k}$ are the boundary values $x(t_0)$, $x(t_0 + T)$, $x(t_0 + 2T)$, $x(t_0 + 3T)$, $x(t_0 + 4T)$, $x(t_0 + 5T)$, and $x(t_0 + 6T)$.

$$\gamma_{k} = \sum_{n=0}^{N-1} \phi(t)$$

are vectors of dimensions $M = 4 \times 2K$. The first four elements

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of $y$, are the values of $X$ and $z$, at the boundaries of $[0, T]$, the remaining elements are the coefficients of the Fourier-type series. Vector $y_t$ can be viewed as a time-independent state parameter vector for $y$, since $y$ characterizes the trajectory of the $n$-th state variable over the time interval $[0, T]$. (Note that the first element is a given initial condition.) The ultimate objective is to determine the optimal state parameters for all state variables that minimize the performance index. This goal is achieved by first relating the state vector, its rate and the control vector to a state parameter vector for $y(t)$.

The state vector containing the $N$ state variables can be written as

$$ y(t) = C(t)z $$

(16)

where

$$ C(t) = \begin{bmatrix} e^t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^t \end{bmatrix} $$

Vector $z$, the state parameter vector for $y$, is a column vector of dimension $NM$. Matrix $C$ is a time-dependent matrix of dimension $N \times NM$. By direct differentiation, the state rate vector can be written as

$$ \dot{y}(t) = D(t)z $$

(18)

where

$$ D(t) = \begin{bmatrix} e^t & 0 \\ \vdots & \ddots \\ 0 & \cdots & e^t \end{bmatrix} $$

The control vector $u(t)$ can also be expressed as a function of $y$. From equation (14)

$$ u(t) = B^{-1}(y(t) + V(z(t))) $$

(20)

where

$$ V(z(t)) = B^{-1}(A(z(t))) $$

(21)

Substituting equations (5) and (14) into equation (20) gives

$$ u(t) = B^{-1}[A^{-1}(y(t) + V(z(t))) + V(z(t))] $$

(22)

Thus, using the Fourier-based state parameterization approach the state vector, state rate vector, and control vector can be represented as functions of the state parameter vector. It is shown in the following sections that by employing this representation LQ problems can be reformulated as QP problems with the elements of the state parameter vector $z$ being the free variables.

**Unconstrained LQ Problems**

The section (i) discusses the conversion process from an unconstrained LQ problem to a QP problem via Fourier-based state parameterization, and (ii) develops an appropriate solution procedure. It is shown that the converted QP problem can be formulated as an unconstrained optimization problem with a quadratic objective function.

**Conversion Process.** The first step in the conversion is to rewrite the performance index as a function of state parameter vector $z$. The terminal state part of the performance index $L_t$ can be written as a function of $y$ by noting the following linear relation for the terminal state vector

$$ y(T) = 0 $$

(23)

where $0$ is a transformation matrix specified according to

$$ \theta = \begin{bmatrix} 1 \cdots 1 \\ m \times (n - 1) \end{bmatrix} \text{ for } n = 1, \ldots, N \quad (24) $$

By substituting equation (23) into equation (3) the cost $J_t$ is

$$ J_t = y^T B^T B y + z^T C(t) y $$

(25)

Similarly, the trajectory part of the performance index $L_t$ can be written as a function of $y$, although the problem is somewhat more complicated. Substituting equation (20) for the control vector into the integrand of equation (6) gives:

$$ \int_{0}^{T} Q(x) + R(y) dy + \int_{0}^{T} \left[ \delta x(t) + \delta y(t) \right] \begin{bmatrix} P_1 & P_2 \end{bmatrix} \left[ \delta x(t) \delta y(t) \right]^T dt $$

(27)

where matrices $P_1$, $P_2$, and $P_3$ and vectors $y_1$ and $y_2$ depend on system parameters and performance index weighting. Subsequently $\delta z$ denotes the noise (white noise). For simplicity, the time-dependent symbol $(t)$ has been omitted in the above equations. By substituting equations (16) and (17) for the state vector and its rate, respectively, into equation (26), the integrand of the performance index can be expressed as a function of parameter vector $z$, i.e.,

$$ \int_{t_0}^{t_1} \int_{t_0}^{t_1} \frac{x^T R_x x + y^T R_y y}{t_0} dt $$

(52)

where

$$ \Lambda = \begin{bmatrix} P_1 & P_2 \end{bmatrix} $$

(33)

In equations (33) and (34), $\delta$ is a Kronecker product of $w$ (see Brower, 1979). Thus, from equation (34), the integral part of the performance index can be expressed as

$$ L_t = \frac{1}{2} \int_{t_0}^{t_1} \Lambda y^T y + y^T \Lambda y + \Sigma \Lambda y $$

(39)

For problems with time-varying system parameters and/or performance index weighting, $P_1$, $P_2$, $P_3$, $y_1$, and $y_2$ are functions of time and the integrals of equations (36) and (37) can be evaluated numerically. For time-invariant problems, $P_1$, $P_2$, $P_3$, $y_1$, and $y_2$ are constants and can be removed from the integrals, enabling the remaining integral parts of $\Lambda$ and $\Sigma$ to be evaluated analytically. That is, for time-invariant problems equations (36) and (37) can be rewritten as

$$ \Lambda = \begin{bmatrix} P_1 & P_2 \end{bmatrix} $$

(56)

The integrals in the brackets of equations (38) and (39) have been evaluated in closed form (Yen, 1949). As a result, the Fourier-based approach is numerically-integration-free in handling time-invariant problems.

In summary, by substituting equations (25) and (26) into equation (1), the performance index $\lambda$ can be written as a quadratic function of the state parameter vector $z$.

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where
\[ L = y^	op y + y^	op y^* \]  

(40)

\[ G - G^* = i\beta \sigma_0 + \alpha \sigma_0, \quad \alpha = G^* + i\beta \]  

(41),(42)

The optimization problem can thus be viewed as the search for the elements of \( y \), i.e., \( x_{mn} = 1, \ldots, N, m = 1, \ldots, M \), that minimize the performance index of equation (40) subject to the equality constraints

\[ x_{mn} = x_{mn}^{*} \text{ for } n = 1, \ldots, N \]  

(43)

representing the initial conditions.

Solution Procedure. This subsection outlines an approach for solving the equality constrained QP problem (outlined above) by converting it into an unconstrained QP problem. To accomplish this goal, a new state parameter vector \( x \) is introduced, specified as

\[ x \equiv x_0 \]  

(44)

where

\[ x_0 = [x_{11} \ldots x_{1N} \quad x_{21} \ldots x_{2N}] \]  

(45),(46)

with

\[ x_{11} = [x_{11} x_{12} \ldots x_{1N}] \]  

(47),(48)

\[ x_{21} = [x_{21} x_{22} \ldots x_{2N}] \]  

(49),(50)

\[ a = [a_{1} \ldots a_{N-1} \ldots a_{N}] \]  

(51)

\[ b = [b_{1} \ldots b_{N-1} \ldots b_{N}] \]  

(52)

\[ \Phi \]  

(53)

The two vectors \( x \) and \( y \) are related via a linear transformation

\[ \begin{bmatrix} y \\ x_0 \end{bmatrix} = \Phi \begin{bmatrix} x \end{bmatrix} \]  

(54)

By expanding equation (53), the performance index can be expressed as

\[ L = [x] \begin{bmatrix} a & \Phi \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \frac{1}{2} x_0 \begin{bmatrix} b \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix} \]  

(55)

or equivalently

\[ L = \begin{bmatrix} x_0 \end{bmatrix} \begin{bmatrix} a & \Phi \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \begin{bmatrix} a & \Phi \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix} \begin{bmatrix} a & \Phi \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix} \]  

(56)

The performance index of equation (58) is a quadratic function of \( x_0 \), the unknown part of the state parameter vector. For an unconstrained QP problem, the necessary condition of optimality can be obtained by differentiating the performance index with respect to this unknown state parameter vector. The result is the system of linear algebraic equations

\[ \begin{bmatrix} a & \Phi \end{bmatrix} \begin{bmatrix} a & \Phi \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} a & \Phi \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix} \]  

(57)

from which the unknown vector \( x_0 \) can be solved directly by a linear algebraic solver, such as a Gaussian elimination routine.

If the terminal conditions of the state vector is known, the same solution procedure can be applied. The only modification is to redefine the unknown vector \( x_0 \) as

\[ x_0 = [x^* x^* x^* x^*] \]  

(58)

and the known vector \( x_2 \) as

\[ x_2 = [x_{21} x_{22} \ldots x_{2N}] \]  

(59)

Similarly, problems with fixed initial and/or terminal state vector can be handled.

An interesting feature of equation (59) is that the coefficient matrix of \( x \) is independent of known boundary values \( i.e., x_{mn} \). Thus, for the basic unconstrained QP problem with different boundary values, the coefficient matrix remains the same, the right-hand-side constant vector needs to be re-computed.

Linearly Constrained QP Problems

In this section, the Fourier-based state parameterization approach is used to convert a linearly constrained QP problem to a QP problem. In particular, the system constraints of equation (5) are converted into a system of linear algebraic constraints (60) can be represented as

\[ \text{The approach is to substitute equation (20) into the inequality constraints of equation (5) giving} \]  

\[ F(x) \leq 0 \]  

(60)

\[ \text{using the state parameterization of equations (16) and (17) in equation (60) gives} \]  

\[ G(x) \leq 0 \]  

(61)

\[ \text{The constraints of equation (60) are functions of time. Consequently, equation (60) represents an infinite number of constraints which need to be satisfied along the trajectory. For practicality, these constraints are relaxed to be satisfied only at a finite number of points (usually chosen to be evenly spaced) in time. That is, equation (60) is replaced by a finite number of algebraic inequalities} \]  

\[ G_{ij} \leq 0 \text{ for } i = 1, \ldots, I \]  

(62)

\[ \text{where } I \text{ is the number of sampling points for which the constraints are only to be satisfied. In terms of the alternate state parameter vector } x \text{, equation (67) can be rewritten using equation (55) as} \]  

\[ G^{(x)}_i \leq 0 \text{ for } i = 1, \ldots, I \]  

(63)

\[ \text{where} \]  

\[ G^{(x)}_i = G(x) \Phi \]  

(64)

By decoupling (x) into and \( a = [a_{1} \ldots a_{N-1} \ldots a_{N}] \), the inequality constraints of equation (60) can be written as

\[ G^{(a)}_i \begin{bmatrix} a_1 \end{bmatrix} \leq 0 \]  

(65)

or equivalently

\[ G^{(a)}_i \begin{bmatrix} a_1 \end{bmatrix} \begin{bmatrix} a_1 \end{bmatrix} \leq \begin{bmatrix} a_1 \end{bmatrix} \begin{bmatrix} a_1 \end{bmatrix} \]  

(66)

Since \( a = [a_{1} \ldots a_{N-1} \ldots a_{N}] \), the corresponding terms can be moved to the right-hand-side of equation (59) giving

\[ G^{(a)}_i \begin{bmatrix} a_1 \end{bmatrix} \begin{bmatrix} a_1 \end{bmatrix} \leq \begin{bmatrix} a_1 \end{bmatrix} \begin{bmatrix} a_1 \end{bmatrix} \]  

(67)

Thus, the system constraints of equation (5) can be approximated by the linear algebraic inequalities of equation (71).

In summary, by applying Fourier-based state parameterization, a linearly constrained QP problem can be converted into a QP problem in which the state equations function of equation (58) is to be minimized without violating the system of linear algebraic constraints of equations (43) and (71).

Fourier-Based Approach for General Linear Systems

The approach presented above is applicable to systems with state and/or control input matrices. For general linear systems, the control influence matrix \( B \), is \( \Delta x \).
matrix where the number of state variables, \( N \), is greater than the number of control variables, \( J \). This section generalizes the Fourier-based approach to the more common case of general linear systems which have fewer control variables than state variables. It is assumed that the rank of \( B \) in (60) is \( J \). To apply the Fourier-based approach, the state-space model of equation (1) is first modified to

\[
\mathbf{x}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)
\]

(72)

and

\[
\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}(t) = \mathbf{u}(t) = \mathbf{B}(t)\mathbf{u}(t)
\]

(73)

(74)

with the subscripts representing the dimensions of the matrices. By introducing an artificial control vector, \( \mathbf{w}(t) \), the new control input matrix, \( \mathbf{B}_w \), can be inverted enabling the calculation of the control, \( \mathbf{u}_w(t) \), for any given trajectory complementing those of (22). Note that it was guaranteed that \( \mathbf{B}_w \) is invertible if the set \( \mathbf{B}_w \) of \( \mathbf{B} \) is non-negative. However, if the set of \( \mathbf{B}_w \) is singular, the first \( N-J \) columns of \( \mathbf{B}_w \) in equation (77) can always be modified to make it invertible since it has been assumed that \( \mathbf{B} \) has rank \( J \).

In order to predict the optimal solution, the performance index (for the case \( \mathbf{Q} = 0 \), \( \mathbf{R} = 0 \)) is modified to

\[
J = \int L \, dt = \int L(t) \, dt
\]

(75)

where \( L \) is the performance index of the original LQ problem, and \( \mathbf{u} \) is a weighting constant chosen to be a large positive matrix, which is not the solution of the original LQ problem. The advantage of using artificial control variables is that a non-actively controlled system can be converted into an actively controlled system in which the Fourier-state parameterization approach is applicable. The drawback is that the resulting solution will not be, in a strict mathematical sense, the trajectory admissibility requirement (see Tim and Nagharks, 1989) due to the nature of the piecewise non-convex artifical control. However, by employing the penalty function of equation (74), the magnitude and influence of the artificial control variables can be made insidental and the solution of the modified optimal control problem can closely approximate the solution of the original LQ problem.

**Simulation Studies**

In the simulations performed, the solution of the time-invariant LQ problems are obtained by Fourier-based state parameterization and simulation with closed-form optimal solutions from standard numerical algorithms. Example 1 is designed to study the effectiveness of the Fourier-based approach to the unconstrained LQ problems. Example 2 and 3 are used to study the effectiveness of Fourier-based parameterization in handling linearly constrained LQ problems. In particular, Example 2 considers a LQ problem with a linear state constraint, whereas Example 3 examines a problem with a bounded control variable. In the first example, the Fourier-based approach is compared to the transition matrix approach, which was applied to generate the state and control variables at prespecified evenly-spaced points in time. An overview of the transition matrix approach for unconstrained LQ problems is presented in Appendix A. Additional details can be found in (Sperly, 1986). In the last two examples, the QP solution algorithm developed by Gill and Murray (1977), considered to be one of the most efficient algorithms for QP problems, was implemented to determine the optimal state parameters of the Fourier-based approach.

**Example 1.** This example considers an N-input Nth order time-invariant dynamic system expressed in canonical form

\[
\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)
\]

(76)

where

\[
\begin{bmatrix}
A \quad B \\
0 \quad I
\end{bmatrix}
\]

(77)

The problem is to determine the control vector that minimizes the performance index

\[
J = \int L(\mathbf{x}) \, dt = \int L(\mathbf{x}, \mathbf{u}) \, dt
\]

(78)

A computationally efficient method for solving this unconstrained LQ problem in the transition matrix approach described in Appendix A. The transition matrix approach converts an optimal control problem into a linear TPBP (such as equation (78)). By evaluating the transition matrix of this boundary value problem, the problem can be reformulated as an initial value problem. In this study, the transition matrices were computed numerically using the algorithm presented in (Franklin and Powell, 1971). The results obtained were obtained at 50 equally-spaced points.

To account for the constraints, the problem could also be solved by integrating the Riccati equation. Although the Riccati method provides the optimal solution in closed-form state, and is thus a preferred approach for physical implementation, it is computationally more intensive than the (open-loop) transition matrix approach. Since the design of an optimal LQ controller is often an iterative process, the transition matrix approach is thus a more efficient test for a Riccati equation solver.

In addition to the transition matrix approach, the Fourier-based approach involving a two-term Fourier-type series was used to solve this problem. The integrals of equations (78) and (79) were determined directly via table look-up. The linear algebraic equations (59) representing the condition of optimality were solved using a Gauss-Jordan elimination routine for the optimal state parameter vector. This vector was used in equation (54) to determine the value of the performance index.

Simulation results for \( N = 2, 4, 8 \) are summarized in Table 1 where execution time (in seconds) is used as an index of computational efficiency. The results demonstrate that the Fourier-based approach is both efficient, especially in solving for the optimal control of high order systems, and accurate (i.e., the error of the performance index is always less than 1 percent). In comparison to the transition matrix approach, the Fourier-based method becomes increasingly more efficient for larger \( N \). For \( N = 20 \) the Fourier-based results suggest a 9 percent reduction in execution time. For \( N = 6 \), the Fourier-based method is less efficient than the transition matrix approach, since the time to evaluate the integrals from the table look-up, a fixed component for any system size, is a significant fraction of the overall computation's cost. For high order systems the principal computational cost is due to the solution of the linear algebraic equations.

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Table 1
Summary of simulation results of example 1

<table>
<thead>
<tr>
<th>Transition Matrix Approach</th>
<th>Fourier-Based Approach†</th>
<th>Comparison</th>
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<tr>
<td>Time</td>
<td>Performance Index</td>
<td>DTE</td>
</tr>
<tr>
<td>2</td>
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</tr>
<tr>
<td>1024</td>
<td>106563.281</td>
<td>193.34</td>
</tr>
</tbody>
</table>

*Best single worst-case Fourier-based system
†Worse performance index of Fourier-based approach relative to performance index value of transition matrix approach.

For satisfactory accuracy, the Fourier-based approach is robust in generating the system impulse for high-order systems. To study robustness, simulations were conducted for the high-order cases \( N = 20, 21, \ldots, 30 \) using the transition matrix and Fourier-based approaches. The results shown in Figure 2 suggest that the transition matrix approach becomes unstable for such high order systems. The figure shows a measure of the relative error of the performance index \( \log \left( 1\Delta tL \right) \) where \( \Delta t \) is the absolute difference of the values of the performance index from the Fourier-based and transition matrix solutions as a function of \( N \). In the Fourier-based approach the relative error is roughly constant as the system order becomes high. In contrast, in the transition matrix approach, the relative error increases (candy-wise) with system order and becomes quite significant. For example, for \( N = 23 \), the relative error measure \( \log \left( 1\Delta tL \right) \) is larger than zero, implying that the relative error of the performance index predicted by the transition matrix approach exceeds 100 percent. The error increases dramatically with system order, indicating that a numerical instability problem has been encountered. This problem is caused predominantly by the error in computing the state transition matrix of the TPBPV equation (A-9). There does not seem to be a computationally efficient solution approach to overcome this numerical difficulty (Bolder and Loan, 1978).

In summary, the Fourier-based approach offers advantages in terms of computational efficiency and numerical robustness relative to the transition matrix approach.

Example 2. This example, adapted from (Ewushenko, 1985, p. 438), considers a time-invariant LQ problem with a time-varying state cost matrix. The system involving two state variables and a single control variable is described by

\[
\begin{align*}
\dot{x}_1 &= 0 \quad x_1(0) = 0 \\
\dot{x}_2 &= -1 \quad x_2(0) = 0
\end{align*}
\]

It is required to find the solution that minimizes the performance index

\[ J = \int \left( s^T \dot{x} + 0.005x^T \dot{x} \right) dt \]

without violating the constraint \( s(x) \leq 5 \) at all times.

where \( s(x) = s \left( x - 0.5 \right)^2 - 0.5 \)

In (Ewushenko, 1985), this problem was solved using a

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Table 2. Summary of Insulation Results for Example 2

<table>
<thead>
<tr>
<th>K</th>
<th>Performance Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1348</td>
</tr>
<tr>
<td>2</td>
<td>0.1784</td>
</tr>
<tr>
<td>3</td>
<td>0.1915</td>
</tr>
<tr>
<td>6</td>
<td>0.1705</td>
</tr>
<tr>
<td>7</td>
<td>0.1759</td>
</tr>
<tr>
<td>8</td>
<td>0.1702</td>
</tr>
<tr>
<td>9</td>
<td>0.1703</td>
</tr>
</tbody>
</table>

Example 3. This example, adapted from (Leondes and Wu, 1978), considers a single input second order system

\[
\begin{align*}
&x_1 = 0.1x_1 + 0.1x_2 + u_0, \\
&x_2 = -1 \cdot x_2 + u_1, \\
&y = x_1 + u_0
\end{align*}
\]

The performance index is

\[
J = \int_0^\infty (x_1^2 + x_2^2) dt
\]

The optimal solution, as computed by Leondes and Wu (1978), has a bang-bang input, i.e.,

\[
\begin{align*}
&u_1 = 0 \text{ for } 0 \leq t < 1.25, \\
&u_1 = 0.8 \text{ for } 1.25 \leq t < 5.0
\end{align*}
\]

The corresponding value of the performance index is 5.600.

This problem was first solved using the Fourier-based approach for a general linear system with a penalty weighting coefficient of \( w = 10 \). The values of the performance index obtained by using a three term Fourier-series are tabulated in Table 2. The four optimal solutions generated by the Fourier-based approach converge to the optimal bang-bang solution as the number of terms of the Fourier-series increases. However, the speed of convergence is quite slow (with a three term Fourier-series the error in the performance index is 47 percent; with a nine term series the error is 9 percent).

The phenomenon of slow-convergence is also evident in the control variable histories. The histories obtained using three, six, and nine term Fourier-series are plotted in Fig. 5(d). The bang-bang optimal solution is also provided in this figure. The
the optimal and Fourier-based solutions can be observed in the neighborhood of the finite jump.

One remedy of this situation is to generalize the "single" segment Fourier-based approach developed in this paper to a "multiple" segment Fourier-based approach which employs spline-like Fourier-based approximations over the trajectory. The idea is to first estimate the location of the instantaneous jump by using the single segment Fourier-based approach, and then approximate each continuous part of the trajectory by a unique Fourier-based representation. For instance, from the results of Fig. 5(a), the time interval [0, 5] is divided into [0, 1.3] and [1.3, 5], each of which is then represented by a unique three-term Fourier-based approximation. Additionally, equality constraints are introduced to ensure the continuity of the state variable response between the two segments. The resulting control variable response is plotted in Fig. 5(b).

The multiple segment Fourier-based solution of Fig. 5(b) simulates the finite jump with greater accuracy than the single segment solution of Fig. 5(a). The performance index of the multiple segment solution is 5.82 which has a 3 percent error compared to Lorenz and Woh's optimal value (and is less than the single segment solutions listed in Table 3). Judged from the value of the performance index, the quality of the Fourier-based solution is more sensitive to the changes at the finite jump and least sensitive to the deviation from the optimal solution \(e^4 < e^5\). Since the performance index is a function only of the state variables, this claim can be verified by examining the response of the state variables, shown in Fig. 5(c). The state variable trajectories of the optimal and Fourier-based solutions are close to agreement. Thus, despite the seemingly poor prediction of the control variable for \(e^4 < e^5\), the state variable histories as well as the value of the performance index are determined satisfactorily. In summary, by employing a multiple segment Fourier-based approach, accurate near optimal solutions of bang-bang control problems can be obtained.

Due to its mathematical complexities, the methodology of the multiple segment Fourier-based approach is not developed in this paper. For applications of the multiple segment approach to unconstrained LQ problems, readers are referred to (Yen, 1980; Yen and Neurauter, 1989).

Discussion

An advantage of a state parameterization approach, such as the Fourier-based approach, is that it characterizes the optimal trajectory which, in theory, consists of an infinite number of points by a relatively small number of state parameters. As optimal control problems can thus be converted into an algebraic optimization program (i.e., a MP problem). In general, the corresponding computations are usually much less complicated than those involved in standard optimal control solvers.

For unconstrained LQ problems, the performance index, which initially is written as a quadratic functional, is converted into a quadratic function. By differentiating this quadratic function with respect to the free parameters, the necessary condition of optimality is derived as a system of linear algebraic equations which can readily be solved. As verified by simulation results, this Fourier-based approach is computationally more efficient than a standard LQ problem solver (based on the transition matrix approach) in handling time-invariant LQ problems.

For linearly constrained LQ problems, the system constraints are relaxed to be satisfied only at a finite number of points (usually equally spaced) in time. Consistently, the linear system constraints are replaced by a finite number of linear algebraic inequalities. The optimal control problem is then converted into a QP problem.

In applying the Fourier-based approach, finite-term Fourier-type series are employed. As a result, the Fourier-based approach...
broach can be classified as a site-optimal (or suboptimal) control approach. The accuracy of the Fourier-based approach can be estimated empirically by increasing the number of terms of the Fourier series. Additional terms can be added to the term by term basis, until the value of the performance index converges, indicating that the optimal solution is closely approximated. For problems with basic characteristics like the Example 7, discontinuities of the optimal solution often cause slow convergence of the Fourier-series approach, as suggested in Example 5, this problem can be overcome by replacing the "single" segment approximation by a "multiple" segment approximation.

Work in progress is generalizing the Fourier-based approach for solving general nonlinear optimal control problems. (Yen, 1988). The underlying idea is similar to sequential quadratic programming, a method which converts a nonlinear programming problem into a sequence of QP problems. Similarly, a nonlinear optimal control problem can be converted into a sequence of linear constrained QP problems, each of which can be solved by an efficient and robust solver, such as the proposed approach.

Conclusion

Based on the idea of near trajectory parameterization, this paper develops a Fourier-based approach for solving unconstrained and linearly constrained QP optimal control problems. It is shown that QP problems can be converted into QP problems. In particular, the necessary condition of optimality for unconstrained QP problems is obtained as a system of linear algebraic equations. Simulation results indicate that the Fourier-based approach is computationally more efficient and numerically more robust than the trajectory matrix approach in handling high order unconstrained QP problems. The results also show that, in many cases, additional terms can be added, at no cost, to the trajectory matrix approach, but at no improvement in accuracy for many linearly constrained QP problems. However, low convergence rates have been observed in applying the Fourier-based approach to the QP problems. This difficulty has been overcome by gradually parameterizing the approach from a single segment approximation to a multiple segment approximation. In summary, the Fourier-based near trajectory parameterization approach appears to be an effective and general computational tool for solving linearly constrained QP problems.

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APPENDIX

Transition Move Approach for Unconstrained QP Problems

Consider the QP problem that minimizes

\[ L = \frac{1}{2} (\mathbf{x}^T \mathbf{S} \mathbf{x}) + \mathbf{d}^T \mathbf{x} \]

subject to

\[ H (\mathbf{x}) \leq 0 \]

For simplicity, consider a system with linear terms of the control and state vectors which has been optimized from the performance index. The order of the system is assumed to be N. In the transition move approach, it is assumed that the state at the current time, x, is given as follows:

\[ x = x_f + \mathbf{A} x_{f-1} + \mathbf{B} u_f \]

where \( x_f \) can be viewed as a Lagrange multiplier vector whose
elements are often called control variables. It can be shown that the necessary conditions of optimality are:

\[
\frac{\partial J}{\partial \lambda_i} = A_i x + B_i u, \quad x(0) = x_0
\]  
(A-4)

\[
\frac{\partial J}{\partial \lambda} = -B_i T_i \lambda + B_i H_i(T),
\]  
(A-5)

\[
0 = \frac{\partial H}{\partial \lambda} = R_0 + B_i \lambda
\]  
(A-6)

Equation (A-6) can be solved for the optimal control giving

\[
u = -B_i^{-1}R_0 - B_i^{-1}B_i \lambda
\]  
(A-7)

Substituting equation (A-7) into equation (A-4) yields

\[\dot{x} = Ax + BR^{-1}B^T \lambda
\]  
(A-8)

Combining equations (A-7) and (A-8) gives a TPBVVP that consists of 2N linear homogeneous differential equations.

\[
[\dot{\lambda}] = A_0 - B R^{-1} B^T [\lambda] = [\lambda](0) = \lambda(T) = \lambda(T_0) \quad (A-9)
\]

This system of equations is often called the Hamiltonian system. Its solution has the following form

\[
[\lambda(0)] = [\phi(T_0)] [\lambda(0)]
\]  
(A-10)

where \( \phi \) is the transition matrix of the Hamiltonian system.

By setting \( T_i = T \) and \( s_i = 0 \), equation (A-10) gives

\[\lambda(T) = [\phi(T_0)] [\lambda(0)]
\]  
(A-11)

With the terminal condition \( \lambda(T) = H(T) \) given by equation (A-9), \( \lambda(0) \) can be determined from equation (A-11) as

\[\lambda(0) = [\lambda(T)] = [\lambda(T_0)]
\]  
(A-12)

where

\[K(T) = [\phi(T_0)] [H(T)] [\phi(T_0)]^T
\]  
(A-13)

The Hamiltonian system of equation (A-9) can thus be viewed as an initial value problem. Using equation (A-10), the solution of this initial value problem can be formulated as

\[\lambda(T) = [\phi(T_0)] [\lambda(0)] + [\phi(T)] [\lambda(0)]
\]  
(A-14)

where \( P \) is the number of equally spaced points for which the solution is required and \( T_i = T/P \). Note that for time-invariant problems, the transition matrix \( [\phi(T)] = [\phi(T)] \) is independent of \( T \) and is only a function of \( T \). A solution approach based on equation (A-14) is computationally much more efficient in general than solving equation (A-11) using numerical integration-based differential equation solvers such as Runge-Kutta methods. The corresponding optimal control is can be computed from equation (A-7). Using this transition-matrix approach, it can also be shown that the corresponding performance index value is

\[L_{\text{optimal}} = \frac{1}{2} x^T(T_0) K(T_0) x(T_0)
\]  
(A-15)

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