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MULTIPLE-SEGMENT FOURIER-BASED APPROACH FOR LINEAR QUADRATIC OPTIMAL CONTROL

RP-6-8

V. Yen
M. Nagurka
Department of Mechanical Engineering
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

A Fourier-based state parameterization approach for determining the near optimal trajectories of linear time-invariant dynamic systems with quadratic performance indices has been developed. The necessary condition of optimality is derived as a system of linear algebraic equations in terms of free boundary values and Fourier coefficients. In contrast to earlier work in which the state trajectory was represented by a single segment Fourier-type approximation, here the use of multiple segment approximations is developed. Simulation results show that the single segment Fourier-based approach is faster than standard transition-matrix and Riccati-based approaches. The multiple segment Fourier-based approach is numerically more robust than the transition-matrix approach and computationally more efficient than Riccati-based approaches in solving linear quadratic optimal control problems. Furthermore, compared to the single segment Fourier-based approach, the multiple segment implementation improves the accuracy of the near optimal solution for highly responsive dynamic systems.

Introduction

Linear quadratic (LQ) optimal control problems have been treated extensively in the control literature.[1-4] Typically, such problems are converted to two-point boundary-value problems (TPBVPs) using the calculus of variations. TPBVPs are then reformulated into initial value problems using a transition matrix approach for open-loop solutions or into terminal value problems using a Riccati-based approach for closed-loop solutions.[5]

In contrast to these variational methods, mathematical programming techniques represent a distinct approach toward the solution of linear and nonlinear optimal control problems. In general, these techniques convert an optimal control problem into an algebraic optimization problem where a near optimal (or suboptimal) solution can be obtained via algebraic optimization. Work done in this area prior to 1970 is summarized by Tabak[6]. A more recent survey is found in Kraft[7]. Theoretical aspects of solving optimal control problems via mathematical programming are well covered.[8,9]

Based on the idea of mathematical programming, Nagurka and Yen[10] developed a Fourier-based method for solving optimal control problems. Unlike typical mathematical programming algorithms for optimal control problems where control variables are characterized by their values at a finite number of points, the Fourier-based approach approximates each state variable by a Fourier-type series superimposed on a polynomial. The actual number of free variables, which is dependent on the number of terms of the Fourier-type series, is usually significantly reduced. Although this single segment Fourier-based approach is computationally very efficient and accurate for most optimal control problems, the near-optimal solution has slow convergence for problems with rapidly changing response characteristics, such as bang-bang control problems.

The Fourier-based approach was specialized by Yen and Nagurka[11] to unconstrained time-invariant LQ systems where the the condition of optimality is formulated as a system of linear algebraic equations. To improve convergence for problems with fast time response, this paper generalizes the approach for LQ problems from a single segment Fourier-type approximation to a multiple segment approximation. That is, the state trajectory is divided into a number of time intervals each of which is approximated by the sum of a polynomial and a Fourier-type series. The results of simulation studies presented here demonstrate that most optimal control problems (Examples 1 and 2) can be handled using a single segment Fourier-based approach. For problems (Example 3) with high speed of response, the multiple segment approach is more effective.

Methodology

This paper considers linear, time-invariant systems described by the state space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

with known initial condition $\mathbf{x}(0) = \mathbf{x}_o$ where $\mathbf{x}$ is an $N \times 1$ state vector, $\mathbf{u}$ is an $J \times 1$ control vector, $\mathbf{A}$ is an $N \times N$ system matrix, and $\mathbf{B}$ is an $N \times J$ control matrix. In the derivation that follows, it is assumed that $J = N$, *i.e.*, the number of control variables is equal to the number of state variables. (Example 2 addresses the case $J < N$.) Furthermore, it is assumed that the control matrix $\mathbf{B}$ is invertible. As a result, every state variable can be "actively" controlled.

The design goal is to find the optimal control $\mathbf{u}(t)$ and the corresponding state trajectory $\mathbf{x}(t)$ in the total time interval $[0, T]$ that minimizes the quadratic performance index

$$L = L_1 + L_2 \tag{2}$$

where

$$L_1 = \mathbf{x}^T(T)\mathbf{H}\mathbf{x}(T) \ , \quad L_2 = \int_o^T \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \tag{3),(4}$$

Matrices $\mathbf{H}$ and $\mathbf{Q}$ are real, symmetric, and positive-semidefinite and matrix $\mathbf{R}$ is real, symmetric, and positive-definite. Superscript T denotes transpose and T (italic) represents the terminal time. It is assumed the state and control vectors are not bounded and the final time T is fixed.

The first step is to divide $[0, T]$ into I intervals $[t_o, t_1]$, $[t_1, t_2]$, . . ., $[t_{I-1}, t_I]$ where $t_o = 0$ and $t_I = T$. (Later in this paper it is shown that for many problems $I = 1$, *i.e.*, a single segment approximation is appropriate.) In the time interval $[t_{i-1}, t_i]$ $(i = 1, 2, ..., I)$ the n-th state variable $x_n(t)$ is approximated by the sum of an auxiliary polynomial and a K term Fourier-type series, *i.e.*, for $i = 1, ..., I$, $n = 1, ..., N$

$$x_n(t) = d_{in}(t) + \sum_{k=1}^{K} a_{ink} \cos\left(\frac{2k\pi(t - t_{i-1})}{\Delta t_i}\right) + \sum_{k=1}^{K} b_{ink} \sin\left(\frac{2k\pi(t-t_{i-1})}{\Delta t_i}\right) \tag{5}$$

where

$$d_{in}(t) = d_{ino} + d_{in1}(t-t_{i-1}) + d_{in2}(t-t_{i-1})^2 + d_{jn3}(t-t_{i-1})^3 \tag{6}$$

$$\Delta t_i = t_i - t_{i-1} \tag{7}$$

If $x_{ino}, \dot{x}_{ino}, x_{inT}$, and $\dot{x}_{inT}$ are the values of the state variable x_n and its derivative at the boundaries of the time segment $[t_{i-1}, t_i]$, *i.e.*,

$$x_{ino} = x_n(t_{i-1}) \quad \dot{x}_{ino} = \dot{x}_n(t_{i-1}) \quad x_{inT} = x_n(t_i) \quad \dot{x}_{inT} = \dot{x}_n(t_i) \tag{8a-d}$$

then the four coefficients of the auxiliary polynomial $d_{in}(t)$ can be written as functions of the boundary values of the segment $[t_{i-1}, t_i]$ and the coefficients of the Fourier series.

$$d_{ino} = x_{ino} - \sum_{k=1}^{K} a_{ink} \ , \quad d_{in1} = \dot{x}_{ino} - \frac{2\pi}{\Delta t_i}\sum_{k=1}^{K} k\, b_{ink} \tag{9a,9b}$$

$$d_{in2} = 3\left(x_{inT} - x_{jn0} + 4\pi\sum_{k=1}^{K} k\, b_{ink}\right)(\Delta t_i)^{-2} - 2\left(\dot{x}_{ino} + \dot{x}_{inT}\right)(\Delta t_i)^{-1} \tag{9c}$$

$$d_{in3} = 2\left(x_{inT} - x_{ino} + 2\pi \sum_{k=1}^{K} k\, b_{ink}\right)(\Delta t_i)^3 + (\dot{x}_{ino} + \dot{x}_{inT})(\Delta t_i)^2 \quad (9d)$$

Using Eq. (9), Eq. (6) can be presented in the form

$$x_n(t) = \rho_{i1}x_{ino} + \rho_{i2}\dot{x}_{ino} + \rho_{i3}x_{inT} + \rho_{i4}\dot{x}_{inT} + \sum_{k=1}^{K}(\alpha_{ik}a_{ink} + \beta_{ik}b_{ink}) \quad (10)$$

over the interval of $[t_{i-1},t_i]$, where

$$\rho_{i1} = 1 - 3\tau_i^2 + 2\tau_i^3 \ , \quad \rho_{i2} = (\tau_i - 2\tau_i^2 + \tau_i^3)\Delta t_i \quad (11a,b)$$

$$\rho_{i3} = 3\tau_i^2 - 2\tau_i^3 \ , \quad \rho_{i4} = (-\tau_i^2 + \tau_i^3)\Delta t_i \quad (11c,d)$$

$$\alpha_{ik} = \cos(2k\pi\tau_i) - 1 \ , \ \beta_{ik} = \sin(2k\pi\tau_i) - 2k\pi\tau_i(1 - 3\tau_i + 2\tau_i^2) \quad (11e,f)$$

with

$$\tau_i = (t - t_{i-1})/\Delta t_i \quad (12)$$

The terms $\rho_{i1}, \dots, \rho_{i4}, \alpha_{ik}, \beta_{ik}$ are functions of time t. Eq. (10) can be written in compact form as

$$x_n(t) = \rho_i^T(t)\, \mathbf{y}_{in} \quad for\ t_{i-1} \le t \le t_i \quad (13)$$

where

$$\rho_i^T(t) = [\ \rho_{i1} \quad \rho_{i2} \quad \rho_{i3} \quad \rho_{i4} \quad \alpha_{i1} \cdots \alpha_{iK} \quad \beta_{i1} \cdots \beta_{iK}\ ] \quad (14)$$

$$\mathbf{y}_{in} = [\ x_{ino} \quad \dot{x}_{ino} \quad x_{inT} \quad \dot{x}_{inT} \quad a_{in1} \cdots a_{inK} \quad b_{in1} \cdots a_{inK}\ ]^T$$
$$= [\ y_{in1} \quad y_{in2} \cdots y_{inM}\ ]^T \quad (15)$$

are vectors of dimension $M = 4 + 2K$. The first four elements of $\mathbf{y}_{in}$ are the values of x_n and $\dot{x}_n$ at the boundary of $[t_{i-1}, t_i]$; the remaining elements are the coefficients of the Fourier-type series. Vector $\mathbf{y}_{in}$ can be viewed as a state parameter vector which characterizes the actual trajectory of x_n over the time interval $[t_{i-1}, t_i]$. The design goal is to search for the optimal parameter values such that the performance index is minimized. To achieve this goal the state vector and its rate are first written as functions of the state parameters.

The state vector $\mathbf{x}(t)$ can now be written as

$$\mathbf{x}(t) = \bar{\rho}_i(t)\mathbf{Y}_i \quad for\ t_{i-1} \le t \le t_i \quad (16)$$

where

$$\bar{\rho}_i = \begin{bmatrix} \rho_i^T & 0 & \cdots & 0 \\ 0 & \rho_i^T & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \rho_i^T \end{bmatrix}, \quad \mathbf{Y}_i = \begin{bmatrix} \mathbf{y}_{i1} \\ \mathbf{y}_{i2} \\ \vdots \\ \mathbf{y}_{iN} \end{bmatrix} = \begin{bmatrix} [y_{i11} \cdots y_{i1M}]^T \\ [y_{i21} \cdots y_{i2M}]^T \\ \vdots \\ [y_{iN1} \cdots y_{iNM}]^T \end{bmatrix} \quad (17a,b)$$

By direct differentiation of Eq. (16), $\dot{\mathbf{x}}(t)$ can be written as:

$$\dot{\mathbf{x}}(t) = \bar{\sigma}_i(t)\mathbf{Y}_i \quad for\ t_{i-1} \le t \le t_i \quad (18)$$

where

$$\bar{\sigma}_i(t) = \dot{\bar{\rho}}_i(t) \quad (19)$$

The next step is to convert the performance index into a function of state parameter vectors $\mathbf{Y}_i$. First, the terminal state vector can be represented as:

$$\mathbf{x}(T) = \Theta\mathbf{Y}_I \quad (20)$$

where Θ is a transformation matrix with elements

$$\theta_{nm} = \begin{cases} 1 & m = (n-1)M + 3 \quad for \quad n = 1, \dots, N \\ 0 & otherwise \end{cases} \quad (21)$$

Substituting Eq. (20) into Eq. (3) gives

$$L_1 = \mathbf{Y}_I^T(\Theta^T\mathbf{H}\Theta)\mathbf{Y}_I^T \quad (22)$$

Similarly, the performance index L_2 can be written as a function of $\mathbf{Y}_i$. First note that since it is assumed that $\mathbf{B}^{-1}$ exists, Eq. (1) can be rewritten as:

$$\mathbf{u} = \mathbf{B}^{-1}\dot{\mathbf{x}} - \mathbf{B}^{-1}\mathbf{A}\mathbf{x} \quad (23)$$

Substituting Eq. (23) into the integrand of Eq. (4) gives:

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} = \mathbf{x}^T\mathbf{F}_1\mathbf{x} + \dot{\mathbf{x}}^T\mathbf{F}_2\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{F}_3\mathbf{x} \quad (24)$$

where

$$\mathbf{F}_1 = \mathbf{Q} + (\mathbf{B}^{-1}\mathbf{A})^T\mathbf{R}(\mathbf{B}^{-1}\mathbf{A}),\ \mathbf{F}_2 = (\mathbf{B}^{-1})^T\mathbf{R}\mathbf{B}^{1} \quad (25a,b)$$

$$\mathbf{F}_3 = -2(\mathbf{B}^{-1})^T\mathbf{R}\mathbf{B}^{-1}\mathbf{A} \quad (25c)$$

are constant matrices that depend on system parameters and performance index weighting. By substituting Eqs. (16) and (18) into Eq. (24), the integrand of the performance index can be expressed as a function of parameter vector $\mathbf{Y}_i$ such that

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} = \mathbf{Y}_i^T\Lambda_i\mathbf{Y}_i \quad for\ t_{i-1} \le t \le t_{i-1} \quad (26)$$

where

$$\Lambda_i = \bar{\rho}_i^T\mathbf{F}_1\bar{\rho}_i + \bar{\sigma}_i^T\mathbf{F}_2\bar{\sigma}_i + \bar{\sigma}_i^T\mathbf{F}_3\bar{\rho}_i = \rho_i\rho_i^T \otimes \mathbf{F}_1 + \sigma_i\sigma_i^T \otimes \mathbf{F}_2 + \sigma_i\rho_i^T \otimes \mathbf{F}_3 \quad (27)$$

The elements of ρ_i and σ_i are functions of time t and time interval Δt_i. In Eq. (27), $\otimes$ is a Kronecker product sign.

Using the results of Eq. (26), the performance index L_2 can be expressed as

$$L_2 = \sum_{i=1}^{I}\int_{t_{i-1}}^{t_i} \mathbf{Y}_i^T\Lambda_i\mathbf{Y}_i\, dt = \sum_{i=1}^{I}\mathbf{Y}_i^T\hat{\Lambda}_i\mathbf{Y}_i \quad (28)$$

where

$$\hat{\Lambda}_i = \int_{t_{i-1}}^{t_i} \Lambda_i\, dt \quad (29)$$

Upon substituting Eq. (27) into Eq. (29), for time invariant problems, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ can be removed from the integral, and the remaining integral parts of $\hat{\Lambda}_i$, which are independent of system and performance index parameters, can be evaluated analytically. These evaluations have been summarized in integral tables for the products (and cross-products) of the elements of ρ_i and σ_i. The availability of such integral tables makes the approach numerically integration-free and thus computationally efficient.

By substituting Eqs. (22) and (28) into Eq. (2), the performance index can be written as a quadratic function:

$$L = \sum_{i=1}^{I-1}\mathbf{Y}_i^T\hat{\Lambda}_i\mathbf{Y}_i + \mathbf{Y}_I^T(\Theta^T\mathbf{H}\Theta + \hat{\Lambda}_I)\mathbf{Y}_I \quad (30)$$

In minimizing this converted performance index, there are two types of constraints that must be satisfied. The first set of constraints refers to the given initial conditions and can be expressed as:

$$y_{1n1} = x_{no} \quad for\ n = 1, \cdots, N \quad (31)$$

where x_{no} is the initial value of the state variable x_n. The second set of constraints refers to the continuity requirements. That is, to ensure continuity between segments, it is required that:

$$x_{(i-1)nT} = x_{ino},\ \dot{x}_{(i-1)nT} = \dot{x}_{ino} \quad for\ i = 1, \cdots, I,\quad n = 1, \cdots, N \quad (32)$$

These equations are equivalent to

$$y_{(i-1)n3} = y_{in1},\ y_{(i-1)n4} = y_{in2} \quad for\ i = 1, \cdots, I,\quad n = 1, \cdots, N \quad (33)$$

The optimization problem can now be formulated as the search for y_{inm}, $i = 1, \dots, I$, $n = 1, \dots, N$, $m = 1, \dots, M$, that minimizes the performance index of Eq. (30) subject to the equality constraints of Eqs. (32) and (33). In particular, by substituting Eqs. (32)-(33) into Eq. (30), this problem can be converted into a unconstrained optimization problem with a quadratic function. Consequently, the necessary condition for optimality can be established by equating to zero the derivatives of the converted performance index with respect to the free variables. Detailed illustrations of such a solution technique can be found in Boot[12].

Problems with fixed terminal states can be formulated exactly as Eqs. (30)-(33) with an additional set of equality constraints

$$y_{fn1} = x_{nT} \quad for \;\; n = 1, \cdots, N \tag{34}$$

where x_{nT} is the prespecified value of the state variables x_n at $t = T$. Problems with linearly constrained terminal states can also be treated similarly.

Simulation Studies

For the simulation studies reported here, the Fourier-based approach and a transition matrix approach or Riccati-based approach were applied to generate the state and control variables at prespecified equally-spaced points in time for LQ problems. To verify accuracy, the values of the performance index from the standard approach and the Fourier-based approach were compared. The time (in *sec*) required to execute the program was recorded for each simulation and was used as an index of the computational efficiency. The computer programs used in the simulations were written in the "C" language and compiled by a Turbo C compiler (Version 2.0). Efforts were made to optimize the speed of the computer codes. The simulations were executed on a 16 MHz NEC 386 PowerMate personal computer with a 16 MHz 80387 coprocessor.

Example 1: The goal of this example is to investigate the effectiveness of the Fourier-based approach for solving high order LQ problems for systems with invertible input matrices. Consider an N input N-th order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\,, \;\; \mathbf{x}^T(0) = [\,1 \;\; 2 \cdots N\,] \tag{35}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \; \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \; \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \cdots \cdots 0 \\ \cdots 0 & 1 & 0 \cdots \cdots \cdots \\ & \ddots & \\ 0 \cdots \cdots 0 & & 1 \\ 1 & -2 \cdots \cdots & (-1)^{(N+1)}N \end{bmatrix}, \; \mathbf{B} = \mathbf{I}_{N \times N} \tag{36}$$

The performance index is

$$L = \mathbf{x}^T(1)\mathbf{H}\mathbf{x}(1) + \int_0^1 \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \tag{37a}$$

$$\mathbf{H} = 10\mathbf{I}_{N \times N}\,, \;\; \mathbf{Q} = \mathbf{R} = \mathbf{I}_{N \times N} \tag{37b}$$

Table I: Summary of Simulation Results of Example 1

	Transition-Matrix Approach		Fourier-Based Approach[a]		Comparison	
N	Performance Index	Time	Performance Index	Time	%Time[b]	Δ%PI[c]
2	5.3591	0.22	5.3591	0.39	177.3	$< 3.7 \times 10^{-5}$
3	44.0044	0.44	44.0045	0.66	150.0	$< 5.9 \times 10^{-4}$
4	44.2499	0.87	44.2504	1.05	120.7	$< 1.1 \times 10^{-3}$
5	164.3776	1.48	164.3884	1.59	107.4	$< 6.6 \times 10^{-3}$
6	153.7563	2.36	153.7622	2.37	100.4	$< 3.9 \times 10^{-3}$
7	399.9883	3.40	400.1103	3.29	96.8	$< 3.1 \times 10^{-2}$
8	373.0219	5.16	373.0597	4.56	88.4	$< 1.1 \times 10^{-2}$
9	788.1612	7.15	788.8568	6.04	84.5	$< 8.9 \times 10^{-2}$
10	741.6136	9.51	741.7737	7.85	82.5	$< 2.2 \times 10^{-2}$
11	1366.9437	12.96	1369.5209	9.99	77.1	$< 1.9 \times 10^{-1}$
12	1299.3828	16.64	1299.8946	12.58	75.6	$< 3.6 \times 10^{-2}$
13	2175.1952	20.81	2182.3431	15.44	74.2	$< 3.3 \times 10^{-1}$
14	2086.3916	26.91	2087.7219	18.73	69.6	$< 6.4 \times 10^{-2}$
15	3252.2758	32.62	3268.4011	22.52	69.0	$< 5.0 \times 10^{-1}$
16	3142.8478	41.08	3145.8080	26.97	65.7	$< 9.5 \times 10^{-2}$

[a] With single segment two-term Fourier-type series
[b] Percent of execution time of Fourier-based approach relative to execution time of transition-matrix approach
[c] Percent difference of performance index of Fourier-based approach relative to performance index of transition-matrix approach

The simulation results for $N = 2, 3, \ldots, 16$ are summarized in Table I assuming a single-segment, two-term Fourier-based approach. The time histories of the state and control variables of the case of $N = 2$ are plotted in Figs. 1a and 1b, respectively. The results demonstrate that a single-segment Fourier-based approximation is sufficiently accurate (always less than 1%) for all cases studied and is especially efficient in solving optimal control problems for high order systems.

Example 2: The goal of this example is to introduce an empirical technique to apply the Fourier-based approach to general linear systems. Here, it is assumed that $J < N$, *i.e.*, the number of control variables is less than the order of the system. The control matrix **B** of Eq. (1) is assumed to be of rank J.

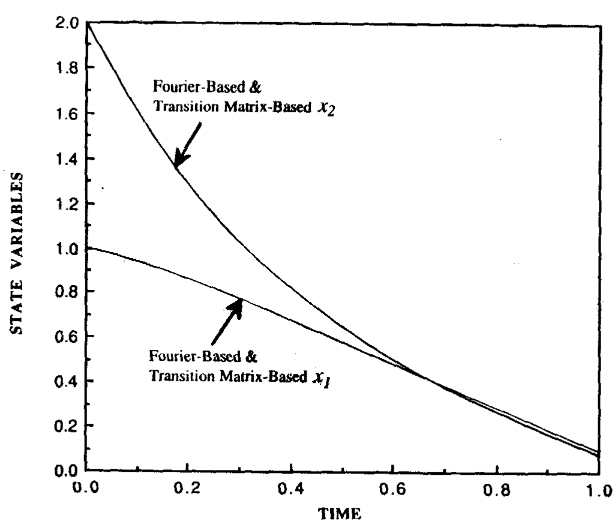

Figure 1a. State Variable Histories for Example 1

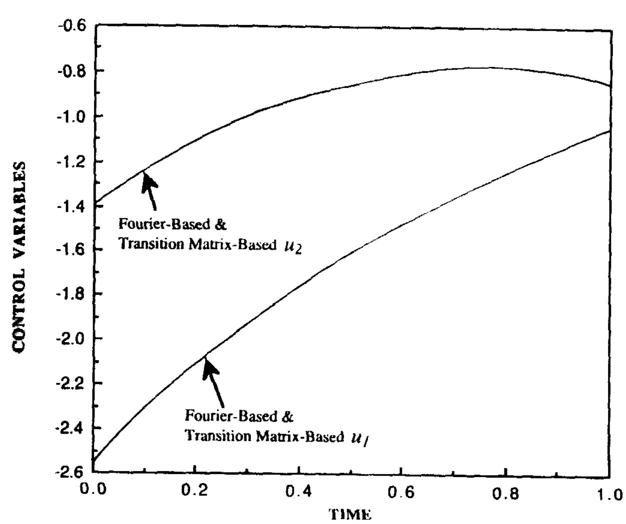

Figure 1b. Control Variable Histories for Example 1

To apply the approach, the state equation is first modified as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}^{*}\mathbf{u}^{*} \tag{38}$$

where

$$\mathbf{u}^{*} = \begin{bmatrix} \hat{\mathbf{u}}_{S\times 1} \\ \mathbf{u}_{J\times 1} \end{bmatrix}, \quad \mathbf{B}^{*} = \begin{bmatrix} \mathbf{I}_{S\times S} & \vdots & \mathbf{B}_{N\times J} \\ 0 & \vdots & \end{bmatrix} \tag{39}$$

with $S = N - J$. $\mathbf{u}$ is the original control vector and $\hat{\mathbf{u}}$ is an artificial control vector which does not physically exist. Since $\mathbf{B}$ is of rank J, without losing generality, it can be assumed that the inverse of $\mathbf{B}^{*}$ exists.

The performance index of the modified system is:

$$L^{*} = L + r\hat{L} \quad \text{where} \quad \hat{L} = \int_0^T \left(\hat{\mathbf{u}}^{T}\hat{\mathbf{u}}\right) dt \tag{40),(41}$$

and L is the performance index of the original system, $\hat{L}$ represents the contribution of the artificial control variables, and r is a positive constant.

To obtain the near-optimal solution for the original problem, r is chosen to be a very large positive constant. As a consequence, solving the optimal solution for the modified system simultaneously minimizes the original performance index and the contribution of the artificial control variables.

The problem specifications of this example are identical to those of Example 1 except that the input matrix $\mathbf{B}$ is now specified as an $N \times 1$ vector as

$$\mathbf{B}^{T} = [\,0 \ldots 0 \ \ 1\,] \tag{42}$$

and the performance index is

$$L = \int_0^1 (\mathbf{x}^{T}\mathbf{x} + u^2)\, dt \tag{43}$$

This problem has been solved using both the transition matrix and a single segment, two-term Fourier-based approach for N = 2, 3, ..., 16. The weighting constant r of the modified performance index was chosen to be 10^5. The simulation results, summarized in Table II, show that the Fourier-based approach is again computationally more efficient in handling high order systems. The time responses of the state and control variables of the case N = 2 are plotted in Figs. 2a and 2b, respectively. These figures show that the solutions from both approaches are hardly distinguishable.

Table II: Summary of Simulation Results of Example 2

	Transition-Matrix Approach		Fourier-Based Approach[a]		Comparison	
N	Performance Index	Time	Performance Index	Time	%Time[b]	Δ%PI[c]
2	27.358	0.22	27.362	0.38	172.8	$< 1.5 \times 10^{-2}$
3	195.033	0.44	195.171	0.66	150.0	$< 6.6 \times 10^{-2}$
4	705.569	0.87	706.255	1.04	135.1	$< 9.8 \times 10^{-2}$
5	1720.550	1.43	1721.381	1.54	107.7	$< 4.9 \times 10^{-2}$
6	3460.001	2.20	3462.970	2.31	105.0	$< 8.6 \times 10^{-2}$
7	6027.753	3.35	6030.865	3.24	96.7	$< 5.2 \times 10^{-2}$
8	9578.606	5.05	9587.778	4.45	88.1	$< 9.6 \times 10^{-2}$
9	14415.109	6.92	14443.140	5.98	86.4	$< 7.8 \times 10^{-2}$
10	20308.134	9.23	20331.694	7.80	84.5	$< 1.2 \times 10^{-1}$
11	28142.031	12.74	28176.018	9.89	77.6	$< 1.3 \times 10^{-1}$
12	36881.498	16.20	36933.743	12.41	76.6	$< 1.5 \times 10^{-1}$
13	48453.432	20.32	48537.693	15.22	74.9	$< 1.8 \times 10^{-1}$
14	60525.689	26.48	60628.971	18.45	69.7	$< 2.8 \times 10^{-1}$
15	76593.643	32.30	76772.657	22.19	68.7	$< 2.4 \times 10^{-1}$
16	92466.982	40.48	92653.710	26.70	66.0	$< 2.1 \times 10^{-1}$

[a]With single segment two-term Fourier-type series
[b]Percent of execution time of Fourier-based approach relative to execution time of transition-matrix approach
[c]Percent difference of performance index of Fourier-based approach relative to performance index of transition-matrix approach

Example 3: The previous examples have used a single segment Fourier-based approximation and the results agree well with the transition matrix solutions. The objective of this example is to explore the use of the multiple segment Fourier-based approximation. This example considers the N = 5 case of Example 1. However, the first diagonal term of the state weighting matrix $\mathbf{Q}$ (*i.e.*, q_{11}) is modified in this example.

Table III summarizes the results of the transition matrix, Riccati equation and Fourier-based approaches (with a two-term Fourier-type series) for q_{11} = 1,10, ..., 10^4. As shown in the table, the transition matrix approach encounters a numerical problem for the case $q_{11} = 10^4$. This problem is caused by the difficulties in numerically evaluating the transition matrix.[13] Table III also shows that the accuracy of the Fourier-

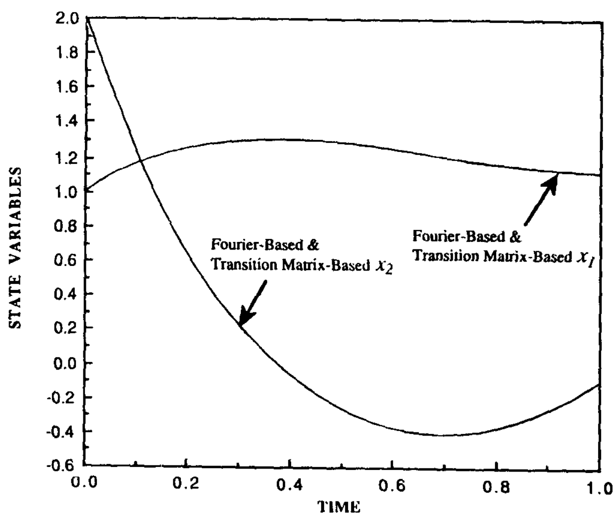

Figure 2a. State Variable Histories for Example 2

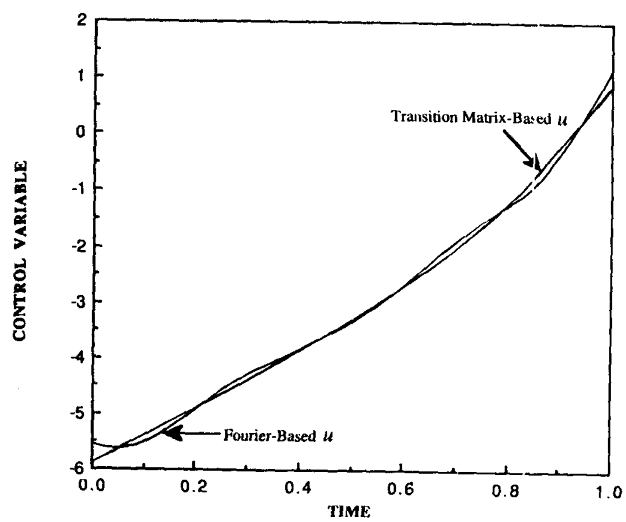

Figure 2b. Control Variable Histories for Example 2

Table III: Summary of Simulation Results of Example 3 with Transition-Matrix, Riccati-Based, and Fourier-Based Approaches

	Transition-Matrix Approach		Riccati-Based Approach		Fourier-Based Approach[a]	
q_{II}	Performance Index	Time	Performance Index	Time	Performance Index	Time
1	164.378	1.48	164.378	24.33	164.388	1.59
10	166.191	1.48	166.191	24.33	166.202	1.59
10^2	171.777	1.48	171.777	24.33	171.816	1.59
10^3	191.760	1.48	191.758	24.33	195.558	1.59
10^4	Unstable		259.241	24.33	377.168	1.59

[a]With single segment two-term Fourier-type series

based approach decreases as q_{II} increases. Among the three methods, only the Riccati equation solver is robust when subject to heavy state weightings. However, the associated computational cost is very high.

Fig. 3a shows the time response of x_I from the Riccati-based and Fourier-based solutions. As shown in the figure, the state variable x_I has a very fast transient response. A response with such a nearly instantaneous shift contains non-negligible high frequency components. As a consequence, it is difficult for a Fourier-type series to achieve a satisfactory approximation with only a few terms. To demonstrate the improvement in accuracy with more terms, Table IV summarizes the simulation results of the Fourier-based approach with two, four, six, eight and ten term series.

Next, a two segment, two-term Fourier-based approximation is applied. Simulation results are summarized in Table V. The results show that by approximating the transient and steady-state responses of state variable x_I with distinct time segments, the Fourier-based approach is highly accurate. The accuracy, however, is influenced by the selection of t_1, the time dividing the two segments. Note that the solution from the single segment Fourier-based approximation (such as shown in Fig. 3a) gives a good indication of where the transient and steady-state response are divided. The time response of x_I plotted in Fig. 3b.

Table IV: Summary of Simulation Results of Example 3 Showing Effect of Different Number of Terms (K) of Fourier-Type Series

K	Performance Index	Time
2	377.168	1.59
4	304.402	4.01
6	279.993	8.46
8	269.731	15.66
10	264.899	26.09

Table V: Summary of Simulation Results of Example 3 with Two Segment Two-Term Fourier-Type Series

t_I	Performance Index[a]
0.05	259.264
0.1	259.478
0.2	262.709
0.3	270.426

[a]Execution time is 5.88 sec

In general, the multiple segment Fourier-based approximation is required only when the time constant of the response is significantly less than the time interval over which the performance index is defined.

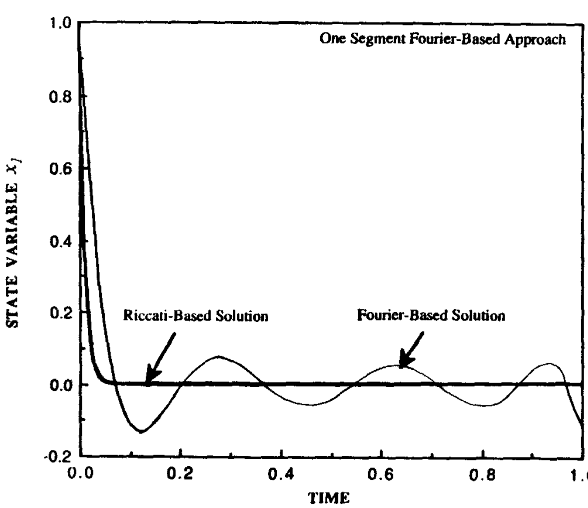

Figure 3a. State Variable Histories for Example 3

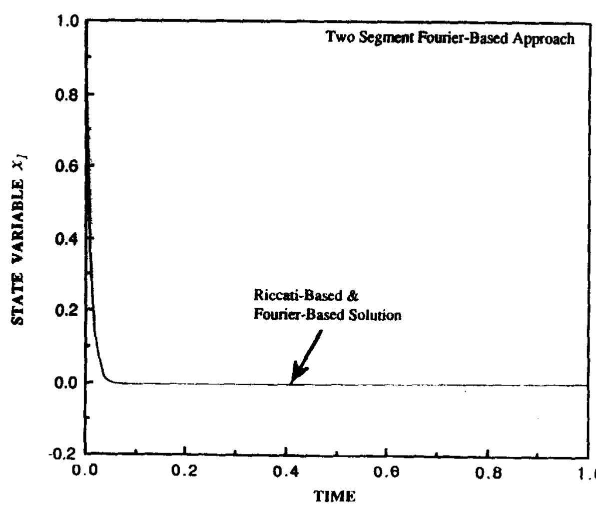

Figure 3b. State Variable Histories for Example 3