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SHORT COMMUNICATIONS

OPTIMAL CONTROL OF LINEARLY CONSTRAINED LINEAR SYSTEMS VIA STATE PARAMETRIZATION

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SUMMARY

This paper presents a general computational tool for determining the near-optimal trajectories of linear, lumped parameter, dynamic systems subjected to linear constraints. In the proposed approach each state variable is approximated by the sum of a third-order polynomial and a finite term Fourier-type series. This enables a linearly constrained optimal control problem to be converted into a linearly constrained mathematical programming problem. Simulation results demonstrate that the approach accurately predicts the optimal performance index and the optimal state and control trajectories.

KEY WORDS Optimal control Mathematical programming State parametrization

INTRODUCTION

Methods for the solution of optimal control problems are treated in depth in many textbooks.[1–4] Typically, the necessary condition of optimality for a constrained optimal control problem is formulated as a two-point boundary-value problem (TPBVP) using Pontryagin's minimum principle. However, the solution of this TPBVP is usually difficult, and in some cases not practical, to obtain. In general, variational methods such as Pontryagin's minimum principle are not effective for solving constrained optimal control problems.

In contrast to variational methods, trajectory parametrization approaches offer an alternative solution strategy. In general, these techniques convert an optimal control problem into a mathematical programming (MP) problem that can be solved for a near-optimal solution via various optimization algorithms. The application of MP methods for solving optimal control problems has been studied by many researchers.[5–14]

The majority of previous trajectory parametrization methods involve control parametrization. In these approaches each control variable is represented by the sum of a sequence of known functions with unknown coefficients serving as control parameters. By representing state variables as functions of control variables, and hence functions of control

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parameters, an optimal control problem can be converted into a MP problem. Numerical integration of the state equations is often required in order to determine the functional relationship between the state and control variables. As a result, converting state-related constraints of an optimal control problem into algebraic inequalities involving a finite number of control parameters is usually a complex process. For the same reason, control parametrization approaches are often intensive computationally and sensitive to numerical errors.

An alternative to control parametrization is to parametrize both the state and control variables, as suggested recently.[14] In combined state and control parametrization an approximation of the system dynamics is generally required to guarantee compatibility between the state and control parameters, i.e. to ensure that the representation of the state and control vectors satisfies the state equations. In addition, the increased number of trajectory parameters tends to increase the computational cost.

State parametrization methods have also been suggested. For example, a Fourier-based state parametrization method[15] has been used to convert a general optimal control problem into a non-linear programming (NP) problem. In this approach each state variable is approximated by a Fourier-type series superimposed on a polynomial. This method was also specialized[16] to handle unconstrained LQ problems. In contrast to control parametrization in which the state equations are used as differential equations, state parametrization treats the state equations as algebraic equations in determining the functional relationship between the state and control vectors. Since numerical integration of the state equations is avoided, state parametrization is usually more efficient than control parametrization. Another advantage of state parametrization is that it can directly handle problems with fixed final states.

This paper promotes a specialized version of the Fourier-based state parametrization approach[15] that converts a linearly constrained optimal control problem into a linearly constrained MP problem. By drawing on the power of well-developed optimization algorithms for linearly constrained MP problems, the proposed approach promises to be an accurate and numerically robust computational tool for determining optimal control trajectories of linearly constrained linear systems.

PROBLEM STATEMENT

Consider a linear dynamic system described by the state space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{1}$$

with known initial condition $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}$ is an $N \times 1$ state vector, $\mathbf{u}$ is a $J \times 1$ control vector, $\mathbf{A}$ is an $N \times N$ system matrix and $\mathbf{B}$ is an $N \times J$ control influence matrix. In the derivation that follows, it is assumed that $J = N$ and $\mathbf{B}$ is invertible, implying that every state variable can be 'actively' controlled. (The case $J < N$ is addressed later.)

The design goal is to find the trajectories of control $\mathbf{u}(t)$ and corresponding state $\mathbf{x}(t)$ in time interval $[0, T]$ that minimize the quadratic performance index

$$L = L_1 + L_2 \tag{2}$$

where

$$L_1 = h(\mathbf{x}(T), T) \tag{3}$$

$$L_2 = \int_0^T g(\mathbf{x}(t), \mathbf{u}(t), t)\, \mathrm{d}t \tag{4}$$

without violating the linear system constraints

$$\mathbf{E}_1(t)\mathbf{x}(t) + \mathbf{E}_2(t)\mathbf{u}(t) \leqslant \mathbf{e}(t) \tag{5}$$

FOURIER-BASED PARAMETRIZATION APPROACH

Fourier-based state parametrization serves as the basis for converting the above time-continuous problem into a finite-dimensional optimization problem. The proposed parametrization can be used to convert the performance index of equation (2) into a function of time-independent state parameters. Additionally, it can be used to replace the system of constraints of equation (5) by a system of linear algebraic inequalities involving only the state parameters.

The basic idea of Fourier-based state parametrization is to divide each state variable trajectory into I intervals, each of which is represented by the sum of a third-order polynomial and a k-term Fourier series. The superposition of the polynomial on the Fourier series increases the speed of convergence and assures differentiability over each interval.[15] As shown in the Appendix, this state parametrization can be cast in the compact form

$$\mathbf{x}(t) = \mathbf{C}_i(t)\mathbf{y}_i \quad \text{for } t_{i-1} \leqslant t \leqslant t_i, \quad i = 1, \ldots, I \tag{6}$$

where $\mathbf{C}_i$ is a matrix of known time-dependent functions (given in the Appendix) and $\mathbf{y}_i$ is a time-dependent state parameter vector that characterizes the response of $\mathbf{x}(t)$ over the ith interval $(t_{i-1} \leqslant t \leqslant t_i)$:

$$\mathbf{y}_i = \begin{bmatrix} \mathbf{y}_{i1} \\ \mathbf{y}_{i2} \\ \vdots \\ \mathbf{y}_{iN} \end{bmatrix} = \begin{bmatrix} [y_{i11} & \cdots & y_{i1M}]^{\mathrm{T}} \\ [y_{i21} & \cdots & y_{i2M}]^{\mathrm{T}} \\ & \vdots & \\ [y_{iN1} & \cdots & y_{iNM}]^{\mathrm{T}} \end{bmatrix} \tag{7}$$

where

$$\begin{aligned} \mathbf{y}_{in} &= [x_{in0} \;\; \dot{x}_{in0} \;\; x_{inT} \;\; \dot{x}_{inT} \;\; a_{in1} \; \ldots \; a_{inK} \;\; b_{in1} \; \ldots \; b_{inK}]^{\mathrm{T}} \\ &= [y_{in1} \;\; y_{in2} \; \ldots \; y_{inM}]^{\mathrm{T}} \end{aligned} \tag{8}$$

consists of the boundary values of x_n and $\dot{x}_n$ (the first four elements of equations (8)) and the coefficients of the Fourier-type series (the last $2K$ elements of equation (8)). Similarly, the state rate vector can be expressed as

$$\dot{\mathbf{x}}(t) = \mathbf{D}_i(t)\mathbf{y}_i \quad \text{for } t_{i-1} \leqslant t \leqslant t_i, \quad i = 1, \ldots, I \tag{9}$$

where

$$\mathbf{D}_i(t) = \dot{\mathbf{C}}_i(t) \tag{10}$$

Assuming that $\mathbf{B}^{-1}$ exists, equation (1) can be rewritten as

$$\mathbf{u}(t) = \mathbf{B}^{-1}(t)\dot{\mathbf{x}}(t) - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{x}(t) \tag{11}$$

Substituting equations (6) and (9) into equation (11) gives

$$\mathbf{u}(t) = (\mathbf{B}^{-1}(t)\mathbf{D}_i(t) - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{C}_i(t))\mathbf{y}_i \quad \text{for } t_{i-1} \leqslant t \leqslant t_i, \quad i = 1, \ldots, I \tag{12}$$

Thus the state, state rate and control can be represented as known functions of the state parameters.

It is now possible to recast the performance index as a function of vectors $\mathbf{y}_i$, $i = 1, \ldots, I$. The first part of the performance index, L_1, can be written as a function of $\mathbf{y}_I$ (the state parameter vector of the last segment) by noting that

$$\mathbf{x}(T) = \mathbf{\Theta y}_I \tag{13}$$

where $\mathbf{\Theta}$ is a transformation matrix with elements

$$\theta_{nm} = \begin{cases} 1, & m = (n-1)M + 3 \quad \text{for } n = 1, \ldots, N \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

Substituting equation (13) into equation (3) gives

$$L_1 = \hat{h}(\mathbf{y}_I, T) \tag{15}$$

Similarly, the second part of the performance index, L_2, can be rewritten by substituting equations (6) and (12) into the integrand of equation (4) to give

$$L_2 = \sum_{i=1}^{I} \int_{t_{i-1}}^{t_i} \hat{g}(\mathbf{y}_i, t)\, \mathrm{d}t \tag{16}$$

From equations (15) and (16) the performance index of equation (2) expressed as a function of the state parameter vector is

$$L = \hat{h}(\mathbf{y}_I, T) + \sum_{i=1}^{I} \int_{t_{i-1}}^{t_i} \hat{g}(\mathbf{y}_i, t)\, \mathrm{d}t \tag{17}$$

The next step is to replace the system constraints of equation (5) by a system of linear algebraic constraints of the state parameters. Substituting equations (6) and (12) into equation (5) gives

$$\mathbf{G}_i(t)\mathbf{y}_i \leqslant \mathbf{e}_i(t) \quad \text{for } t_{i-1} \leqslant t \leqslant t_i, \quad i = 1, \ldots, I \tag{18}$$

where

$$\mathbf{G}_i(t) = (\mathbf{E}_1(t) - \mathbf{E}_2(t)\mathbf{B}^{-1}(t)\mathbf{A}(t))\mathbf{C}_i(t) + \mathbf{E}_2(t)\mathbf{B}^{-1}\mathbf{D}_i(t) \tag{19}$$

Since the constraints of equation (18) are functions of continuous time, they represent an infinite number of constraints that must be satisfied along the trajectory. In order to convert these constraints into a finite number of algebraic inequalities, the constraints are relaxed to be satisfied only at a finite number of points (usually chosen to be equally spaced) in time. Consequently, equation (18) is replaced by

$$\hat{\mathbf{G}}_i\mathbf{y}_i \leqslant \hat{\mathbf{e}}_i \quad \text{for } i = 1, \ldots, I \tag{20}$$

where

$$\hat{\mathbf{G}}_i = \begin{bmatrix} \mathbf{G}_i(t_{i-1}) \\ \mathbf{G}_i(t_{i-1} + \delta t_i) \\ \vdots \\ \mathbf{G}_i(t_{i-1} + (m_i - 1)\,\delta t_i) \\ \mathbf{G}_i(t_i) \end{bmatrix} \tag{21}$$

$$\hat{\mathbf{e}}_i = \begin{bmatrix} \mathbf{e}_i(t_{i-1}) \\ \mathbf{e}_i(t_{i-1} + \delta t_i) \\ \vdots \\ \mathbf{e}_i(t_{i-1} + (m_i - 1)\,\delta t_i) \\ \mathbf{e}_i(t_i) \end{bmatrix} \tag{22}$$

with

$$\delta t_i = \Delta t_i / m_i \tag{23}$$

where m_i is the number of sampling points for the ith segment.

In minimizing the converted performance index, two types of constraints must be satisfied. The first set of constraints guarantees that the given initial conditions are met:

$$y_{1n1} = x_{n0} \quad \text{for } n = 1, \ldots, N \tag{24}$$

where x_{n0} is the initial value for the state variable x_n. The second set of constraints ensures continuity between segments:

$$x_{(i-1)nT} = x_{in0} \quad \text{for } i = 2, \ldots, I, \quad n = 1, \ldots, N \tag{25}$$

$$\dot{x}_{(i-1)nT} = \dot{x}_{in0} \quad \text{for } i = 2, \ldots, I, \quad n = 1, \ldots, N \tag{26}$$

By definition from equation (8), these equations are equivalent to

$$y_{(i-1)n3} = y_{in1} \quad \text{for } i = 2, \ldots, I, \quad n = 1, \ldots, N \tag{27}$$

$$y_{(i-1)n4} = y_{in2} \quad \text{for } i = 2, \ldots, I, \quad n = 1, \ldots, N \tag{28}$$

The optimization problem can now be viewed as the search for y_{inm}, $i = 1, \ldots, I$, $n = 1, \ldots, N$, $m = 1, \ldots, M$, that minimizes the performance index of equation (17) subject to the equality and inequality constraints of equations (20), (24), (27) and (28).

FOURIER-BASED APPROACH FOR GENERAL LINEAR SYSTEMS

The approach presented above is applicable to systems with square and invertible control influence matrices, often called 'actively' controlled systems. This section generalizes the Fourier-based approach to the more common case of linear systems which have fewer control variables than state variables. The system has the linear structure described by equation (1) with $\mathbf{B}$ being an $N \times J$ matrix, where $N > J$. It is assumed that the rank of $\mathbf{B}$ is J.

To apply the Fourier-based approach, the state space model of equation (1) is first modified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}^*(t)\mathbf{u}^*(t) \tag{29}$$

where

$$\mathbf{B}^*(t) = \mathbf{B}^*_{N\times N} = \begin{bmatrix} \mathbf{I}_{(N-J)\times(N-J)} & \mathbf{B}_{N\times J} \\ \mathbf{O}_{J\times(N-J)} & \end{bmatrix} \tag{30}$$

$$\mathbf{u}^*(t) = \mathbf{u}^*_{N\times 1} = \begin{bmatrix} \hat{\mathbf{u}}_{(N-J)\times 1} \\ \mathbf{u}_{J\times 1} \end{bmatrix} \tag{31}$$

and the subscripts represent the dimensions of the matrices. By introducing an artificial control vector $\hat{\mathbf{u}}$, the new control influence matrix $\mathbf{B}^*$ can be inverted, enabling the calculation of the control $\mathbf{u}^*$ for any given trajectory. It can be guaranteed that $\mathbf{B}^*$ is invertible if the last J rows of $\mathbf{B}$ are non-singular. However, if these rows are singular, since $\mathbf{B}$ has rank J, the first $N - J$ columns of $\mathbf{B}^*$ can always be adjusted to ensure that it is invertible.

In order to predict the optimal solution, the performance index is modified to

$$L^* = L + r \int_0^T \hat{\mathbf{u}}^{\mathrm{T}}(t)\hat{\mathbf{u}}(t)\,\mathrm{d}t \tag{32}$$

where L is the performance index of the original optimal control problem (with control $\mathbf{u}^*$) and r is a weighting constant chosen to be a large positive number. The integral term associated with r represents the contribution of the artificial control vector.

By using artificial control variables, a non-actively controlled system can be converted into an actively controlled system to which the Fourier-based state parametrization approach can be applied. The resulting solution will not, in a strict mathematical sense, satisfy the trajectory admissibility requirement owing to the existence of artificial control variables.[16] However, by employing the penalty function approach of equation (32), the magnitude and influence of the artificial control variables can be made small and the solution of the modified optimal control problem can approximate closely the solution of the original optimal control problem.

SIMULATION STUDIES

In this section the solutions of optimal control problems obtained by the Fourier-based approach are compared with known closed-form solutions. Example 1 investigates a minimum fuel problem with bounded control. Example 2 examines a bang–bang control problem with a quadratic performance index and bounded control. In these examples the modified Newton method developed by Gill and Murray,[17,18] considered to be one of the most efficient solution approaches for linearly constrained MP problems, is used to determine the optimal values of the unknown state parameters. Efforts were made to optimize the speed of the computer codes, all of which were written in 'C'. The simulations were executed on a SUN 3/60 workstation.

Example 1

This minimum fuel problem is adapted from Owen[19] (pp. 251–255). The system is described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u \tag{33}$$

with boundary conditions

$$\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},\qquad \begin{bmatrix}x_1(1)\\ x_2(1)\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix} \tag{34a,b}$$

It is required to find the solution that minimizes the performance index

$$L=\int_0^1 |u(t)|\,\mathrm{d}t \tag{35}$$

with a constraint imposed on the control

$$|u(t)|\leqslant p \tag{36}$$

The closed-form optimal solution, derived by Owen,[19] is

$$u(t)=\begin{cases} p, & 0\leqslant t<t_1\\ 0, & t_1<t<t_2\\ -p, & t_2<t\leqslant 1\end{cases} \tag{37}$$

with the switching times

$$t_1=\tfrac{1}{2}\left[1-\sqrt{(1-4/p)}\right],\qquad t_2=\tfrac{1}{2}\left[1+\sqrt{(1-4/p)}\right] \tag{38a,b}$$

A solution of the specified minimum fuel problem exists if and only if $p \geqslant 4$. (Otherwise the switching times are complex.) In this example we choose $p = 5$.

Owing to the absolute sign in the performance index integrand, the linearly constrained MP problem resulting from application of Fourier-based state parametrization is non-differentiable. Optimization algorithms for linearly constrained MP problems typically assume differentiability of the objective function. To overcome this difficulty, an empirical technique is introduced in which the performance index of equation (35) is replaced by

$$L = \int_0^1 (g(u)u)\,\mathrm{d}t \tag{39}$$

where

$$g(u) = \frac{4}{\pi}\left[\sin\left(\frac{\pi u}{\eta}\right) + \sin\left(\frac{3\pi u}{\eta}\right) + \sin\left(\frac{5\pi u}{\eta}\right) + \cdots\right] \tag{40}$$

In this example the number of terms of $g(u)$ is chosen to be 20 and $\eta = 10$.

The optimal control history (i.e. equation (37)) and the solutions obtained using three-, five- and nine-term Fourier-type series are plotted in Figure 1. The corresponding performance index values are summarized in Table I. As shown in both Figure 1 and Table I, the Fourier-

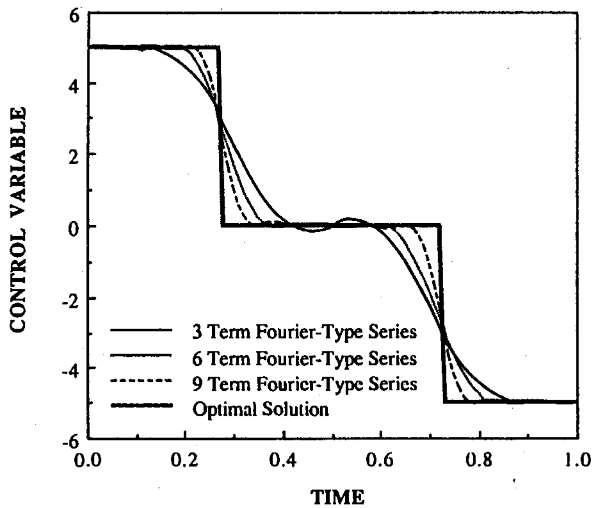

Figure I. Control variable histories for Example 1

Table I. Summary of simulation results of Example 1 using single-segment K-term Fourier-type series (optimal solution is 2·7639)

K	Performance index
3	2·8872
6	2·8211
9	2·7838

based solutions converge to the optimal solution as the number of terms of the Fourier-type series increases. Since the Fourier-based approach assumes continuity throughout the trajectory, significant mismatches are observed in the control variable histories at the switching times. Despite the discrepancies, the Fourier-based approach provides solutions which approximate closely the optimal performance index.

Compared to traditional methods for minimum fuel problems, the Fourier-based approach does not require any specialized analytical or computational effort to determine the switching times. Furthermore, the Fourier-based solution is continuous throughout the trajectory and hence avoids the bang–off–bang characteristic of the optimal solution which is generally not physically implementable.

Example 2

This example, adapted from Leondes and Wu,[20] considers the linear quadratic (LQ) system described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u, \qquad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}0\cdot 167\\ 1\cdot 120\end{bmatrix} \tag{41}$$

with performance index

$$L=\tfrac{1}{2}\int_0^5 (x_1^2+x_2^2)\,\mathrm{d}t \tag{42}$$

As in Example 1, a constraint is imposed on the control. Here

$$|u(t)|\leqslant 0\cdot 8 \tag{43}$$

The optimal solution, as computed by Leondes and Wu,[20] has a bang–bang nature, i.e.

$$u(t)=\begin{cases}-0\cdot 8 & 0\leqslant t<1\cdot 325\\ 0\cdot 8, & 1\cdot 325<t\leqslant 5\cdot 0\end{cases} \tag{44}$$

The corresponding value of the performance index is 5·607.

The control histories using the Fourier-based approach with three-, six- and nine-term Fourier-type series are plotted in Figure 2. Values of the performance index as a function of the number of terms are tabulated in Table II. From Table II and Figure 2 it is seen that the state-parametrized solutions begin to converge to the optimal bang–bang solution as the number of terms of the Fourier-type series increases. However, the speed of convergence is quite slow, since the Fourier-based approach assumes continuity throughout the trajectory and the optimal solution exhibits an instantaneous switch at $t=1\cdot 325$. Consequently, significant discrepancies between the optimal and Fourier-based solutions can be observed in the neighbourhood of the finite jump.

One remedy for this slow convergence is the application of a multiple-segment Fourier-based approach. The idea is to estimate the locations of the instantaneous jumps by using the single-segment Fourier-based approach and then to represent each continuous part of the trajectory by a unique Fourier-based representation. In this example, based on the results of Figure 2, the time interval [0, 5] is divided into two intervals, [0, 1·3] and [1·3, 5·0]. Then a three-term Fourier-based representation is employed for each interval. The resulting performance index value is 6·161, which has a 9·9 true percentage relative error and is better than all the single-segment solutions listed in Table II except the one for nine terms. The corresponding control history is plotted in Figure 3. Since the Fourier-based solution has no discontinuity, it is more likely to be physically implementable.

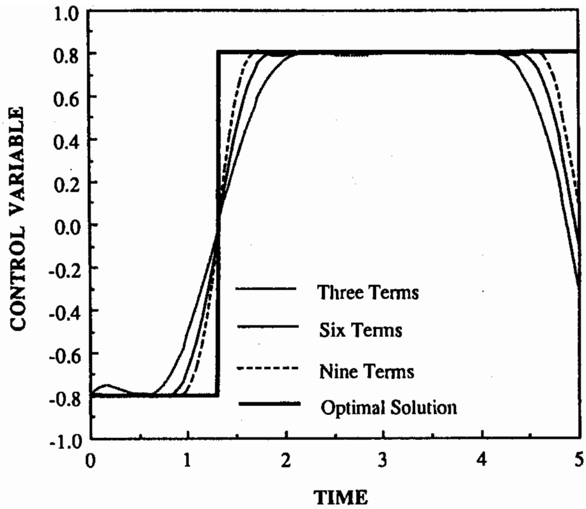

Figure 2. Control variable histories for Example 2 using single-segment Fourier-based approach

Table II. Summary of simulation results of Example 2 using single-segment K-term Fourier-type series (optimal solution is 5·607)

K	Performance index
3	8·357
4	7·832
5	6·909
6	6·548
7	6·467
8	6·238
9	6·092

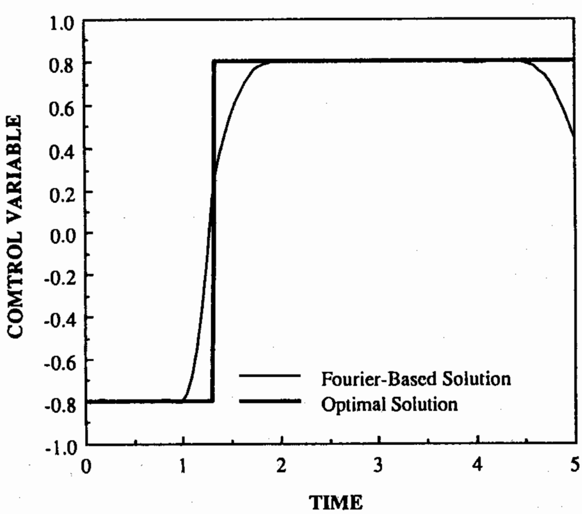

Figure 3. Control variable histories for Example 2 using double-segment three-term Fourier-based approach (with continuity requirement on state rate)

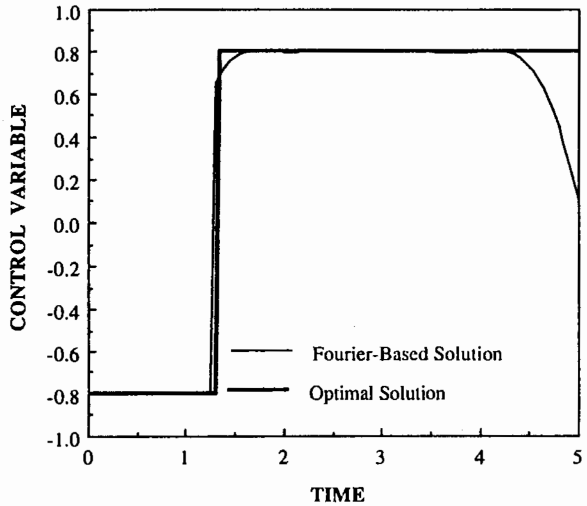

Figure 4. Control variable histories for Example 2 using double-segment three-term Fourier-based approach (without continuity requirement on state rate)

The Fourier-based solution can be improved further by removing the continuity requirement on the state variable rate. By not imposing the equality constraint of equation (28) (or, equivalently, equation (26)), the control variables of the Fourier-based solution can be discontinuous at the point between segments, here $t = 1 \cdot 3$. As a result the Fourier-based solution can more closely simulate the bang–bang characteristic of the optimal solution. By relaxing the equality constraint on the state rate and using the double-segment three-term Fourier-based approach, the performance index value becomes 5·709, which is only 2% larger than the optimal value. The control response of the double-segment solution is plotted in Figure 4. Judging from the value of the performance index, the quality of system performance is very sensitive to the changes at the finite jump and not very sensitive to the deviation from the optimal solution for $4 < t < 5$.

DISCUSSION

This paper presents a Fourier-based state parametrization approach for solving linearly constrained optimal control problems. State parametrization approaches using eigenfunctions other than Fourier-type series, such as Chebyshev polynomials, are possible and have been reported[21] for unconstrained problems. In employing an alternative representation and converting a linearly constrained optimal control problem to an MP problem, it is essential that (i) convergence be guaranteed on the state and state rate vectors and (ii) continuity on the state and control vectors be achieved. The Fourier-based approach was selected since it readily satisfies these requirements and has well-known properties.

The Fourier-based approach, like other trajectory parametrization approaches, characterizes the system response by a relatively small number of parameters and casts the original time-continuous optimal control problem as an MP problem. In the proposed approach the state equations are treated as algebraic equations in establishing the functional relationship between the state and control variables (see equation (11)). In control parametrization the state equations are manipulated as differential equations and numerical integration is often required in order to represent state variables as functions of control variables. These differences are often summarized by viewing state parametrization as being an 'inverse dynamics' approach, whereas control parametrization is based on 'direct dynamics'. As a consequence, as long as problems with trajectory inadmissibility are avoided (where constraints on the control structure prevent an arbitrary representation of the state trajectory from being achieved), state parametrization approaches are generally more robust and efficient than control parametrization approaches. To ensure trajectory admissibility in the proposed approach, artificial control variables are suggested. Although they can increase the computational cost, especially if the state/control variable ratio is high, they enable the control to be represented by the state parameters and guarantee general applicability of the method.

Because finite term Fourier-type series are employed, the Fourier-based approach is a near-optimal (or suboptimal) method. The accuracy can be estimated empirically by increasing the number of terms on a term-by-term basis until the performance index converges to a desired accuracy. For problems with continuous optimal solutions a single segment Fourier-based representation is usually sufficient. For problems with discontinuous optimal solutions (such as Examples 1 and 2) a multiple-segment approximation may help to achieve an accurate near-optimal solution. The Fourier-based approach has the flexibility of maintaining continuity of the near-optimal solution. This may avoid problems with physical implementation often encountered in applying the bang–bang control laws suggested by standard optimal control solution approaches. Another advantage of the Fourier-based approach is that it can easily handle problems with fixed initial and terminal states and state rates, since these boundary values are part of the state parameters.

CONCLUSIONS

Drawing on the idea of state trajectory parametrization, this paper develops a Fourier-based approach for solving linearly constrained optimal control problems with linear dynamics. These optimal control problems are converted to linearly constrained MP problems that are tackled using well-developed optimization algorithms. The results of simulation studies demonstrate that the Fourier-based approach is capable of providing accurate and continuous near-optimal solutions even when the optimal solution is not continuous. The approach promises to be an effective and general computational tool for solving linearly constrained optimal control problems.

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APPENDIX

This appendix provides the analytical foundation of the multiple-segment Fourier-based state parametrization approach. The first step is to divide $[0, T]$ into I intervals $[t_0, t_1]$, $[t_1, t_2], \ldots, [t_{I-1}, t_I]$, where $t_0 = 0$ and $t_I = T$. In time interval $[t_{i-1}, t_i]$ $(i = 1, 2, \ldots, I)$ the nth state variable $x_n(t)$ is approximated by the sum of a third-order polynomial $p_{in}(t)$ and a

K-term Fourier-type series, i.e. for $i = 1, \dots, I$ and $n = 1, \dots, N$,

$$x_n(t) = p_{in}(t) + \sum_{k=1}^{K} a_{ink} \cos\left(\frac{2k\pi(t - t_{i-1})}{\Delta t_i}\right) + \sum_{k=1}^{K} b_{ink} \sin\left(\frac{2k\pi(t - t_{i-1})}{\Delta t_i}\right) \tag{45}$$

where

$$p_{in}(t) = p_{in0} + p_{in1}(t - t_{i-1}) + p_{in2}(t - t_{i-1})^2 + p_{in3}(t - t_{i-1})^3 \tag{46}$$

$$\Delta t_i = t_i - t_{i-1} \tag{47}$$

Compared to a standard Fourier series expansion, this representation assures high speed of convergence and differentiability.[15]

The coefficients of $p_{in}(t)$ can be written as functions of the coefficients of the Fourier series and the values of the state variable x_n and its derivative at the boundaries of the time segment $[t_{i-1}, t_i]$, i.e. x_{in0}, $\dot{x}_{in0}$, x_{inT} and $\dot{x}_{inT}$, where

$$x_{in0} = x_n(t_{i-1}), \qquad \dot{x}_{in0} = \dot{x}_n(t_{i-1}), \qquad x_{inT} = x_n(t_i), \qquad \dot{x}_{inT} = \dot{x}_n(t_i) \tag{48a–d}$$

where $t_0 = 0$ and $t_I = T$.

$$p_{in0} = x_{in0} - \sum_{k=1}^{K} a_{ink}, \qquad p_{in1} = \dot{x}_{in0} - \frac{2\pi}{\Delta t_i} \sum_{k=1}^{K} k b_{ink} \tag{49a,b}$$

$$p_{in2} = 3\left(x_{inT} - x_{in0} + 4\pi \sum_{k=1}^{K} k b_{ink}\right) \Delta t_i^{-2} - 2(\dot{x}_{in0} + \dot{x}_{inT}) \Delta t_i^{-1} \tag{49c}$$

$$p_{in3} = 2\left(x_{inT} - x_{in0} + 2\pi \sum_{k=1}^{K} k b_{ink}\right) \Delta t_i^{-3} + (\dot{x}_{in0} + \dot{x}_{inT}) \Delta t_i^{-2} \tag{49d}$$

Substituting these expressions into equation (45) and rearranging gives

$$x_n(t) = \rho_{i1} x_{in0} + \rho_{i2} \dot{x}_{in0} + \rho_{i3} x_{inT} + \rho_{i4} \dot{x}_{inT} + \sum_{k=1}^{K} (\alpha_{ik} a_{ink} + \beta_{ik} b_{ink}) \tag{50}$$

where

$$\rho_{i1} = 1 - 3\tau_i^2 + 2\tau_i^3, \qquad \rho_{i2} = (\tau_i - 2\tau_i^2 + \tau_i^3)\,\Delta t_i \tag{51a,b}$$

$$\rho_{i3} = 3\tau_i^2 - 2\tau_i^3, \qquad \rho_{i4} = (-\tau_i^2 + \tau_i^3)\,\Delta t_i \tag{51c,d}$$

$$\alpha_{ik} = \cos(2k\pi\tau_i) - 1, \qquad \beta_{ik} = \sin(2k\pi\tau_i) - 2k\pi\tau_i(1 - 3\tau_i + 2\tau_i^2) \tag{51e,f}$$

with

$$\tau_i = (t - t_{i-1})/\Delta t_i \tag{52}$$

Equation (50) can be written in compact form as

$$x_n(t) = \mathbf{c}_i^{\mathrm{T}}(t)\mathbf{y}_{in} \quad \text{for } t_{i-1} \leqslant t \leqslant t_i \tag{53}$$

where

$$\mathbf{c}_i^{\mathrm{T}}(t) = [\rho_{i1}\ \rho_{i2}\ \rho_{i3}\ \rho_{i4}\ \alpha_{i1}\ \dots\ \alpha_{iK}\ \beta_{i1}\ \dots\ \beta_{iK}] \tag{54}$$

$$\begin{aligned}\mathbf{y}_{in} &= [x_{in0}\ \dot{x}_{in0}\ x_{inT}\ \dot{x}_{inT}\ a_{in1}\ \dots\ a_{inK}\ b_{in1}\ \dots\ b_{inK}]^{\mathrm{T}} \\ &= [y_{in1}\ y_{in2}\ \dots\ y_{inM}]^{\mathrm{T}}\end{aligned} \tag{55}$$

are vectors of length $M = 4 + 2K$. The first four elements of $\mathbf{y}_{in}$ are the values of x_n and $\dot{x}_n$

at the boundary of $[t_{i-1}, t_i]$; the last $2K$ elements are the coefficients of the Fourier-type series.

The state vector $\mathbf{x}(t)$ can now be written as

$$\mathbf{x}(t) = \mathbf{C}_i(t)\mathbf{y}_i \quad \text{for } t_{i-1} \leqslant t \leqslant t_i \tag{56}$$

where the state parameter vector $\mathbf{y}_i$ has length $N(4+2K)$ and

$$\mathbf{C}_i = \begin{bmatrix} \mathbf{c}_i^{\mathrm{T}} & & & \mathbf{0} \\ & \mathbf{c}_i^{\mathrm{T}} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{c}_i^{\mathrm{T}} \end{bmatrix} \tag{57}$$

$$\mathbf{y}_i = \begin{bmatrix} \mathbf{y}_{i1} \\ \mathbf{y}_{i2} \\ \vdots \\ \mathbf{y}_{iN} \end{bmatrix} = \begin{bmatrix} [y_{i11} & \cdots & y_{i1M}]^{\mathrm{T}} \\ [y_{i21} & \cdots & y_{i2M}]^{\mathrm{T}} \\ & \vdots & \\ [y_{iN1} & \cdots & y_{iNM}]^{\mathrm{T}} \end{bmatrix} \tag{58}$$

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