OPTIMAL TRAJECTORY PLANNING OF ROBOTIC MANIPULATORS VIA QUASI-LINEARIZATION AND STATE PARAMETERIZATION

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This paper develops a methodology for trajectory planning of robotic manipulators. The trajectory planning problem, cast as an unconstrained nonlinear optimal control problem, is converted into a sequence of linear quadratic (LQ) optimal control problems via quasi-linearization. Each of the LQ problems is solved via Fourier-based state parameterization. In contrast to variational methods and dynamic programming, the approach is computationally efficient and guarantees a near optimal solution that satisfies terminal conditions.

INTRODUCTION

In addition to variational methods and dynamic programming, mathematical programming techniques (including linear, quadratic, and nonlinear programming) offer important means for solving optimal control problems. For example, linear programming has been applied to solve linear optimal control problems involving minimum time [1,2], minimum fuel [3,4], and both [5-7]. Similarly, quadratic programming has been used to solve linear optimal control problems [4,8-10]. General nonlinear optimal control problems have been solved by nonlinear programming techniques [11-12].

Typically, the above methods convert an optimal control problem into an algebraic optimization problem by assuming that the control variables are piecewise continuous. The number of free variables of the resulting optimization problem is usually high, limiting the effectiveness and efficiency of computational algorithms. This is particularly true for systems of high-order and/or with large terminal times.

Efforts have been made to reduce the dimensionality of the converted optimization problem. For example, Yen and Nagurka [13] represented the state variables by the sum of a polynomial and a Fourier-type series, and showed important computational advantages when solving linear quadratic (LQ) problems. Yen and Nagurka [14] also developed computationally efficient algorithms in which the control variables of linear systems were represented by series of orthogonal functions. For nonlinear optimal control problems, Yen and Nagurka [15] employed a Fourier-based approximation of the state variables. The state and control variables have also been expanded in Chebyshev series [16]. Compared to other parameterization approaches, these methods tend to involve fewer free variables in the algebraic optimization problem. However, the speed of convergence is often quite slow for nonlinear problems. Consequently, except in the special case of LQ problems, these mathematical-programming-based methods are computationally intensive for solving optimal control problems.

Due to these difficulties (and the inability of variational methods and dynamic programming to solve high order nonlinear optimal control problems), the optimal control of general robotic manipulators has remained a research challenge. Some specialized problems have been addressed. For example, minimum-time trajectory problems, which have bang-bang control solutions, have been studied extensively [17-24]. Robotic manipulator problems involving general performance indices have been investigated using dynamic programming [25-27]. These approaches circumvent the dimensionality problem by restricting the degrees-of-freedom of the trajectories, thus limiting the quality of the "claimed" optimal solution.

Mathematical programming methods have also been applied to robotic manipulator systems. In these approaches the joint displacements are represented by sets of functions whose optimal coefficient values are determined. In Schmitt, *et al* [28], these coefficients are computed from the necessary conditions of optimality which are a system of nonlinear algebraic equations. In Nagurka and Yen [29], the coefficients are obtained via nonlinear programming. Both approaches are robust in terms of numerical stability, but require substantial computation.

This paper applies a quasi-linearization approach [30] to determine the optimal control of robotic manipulators. The approach employs a second-order approximation of the performance index and converts the nonlinear optimal control problem into a sequence of time-varying LQ problems with fixed terminal states. Fourier-based state parameterization is applied to solve each of these LQ problems. Simulation results indicate that the approach is accurate and computationally efficient.

OPTIMAL CONTROL OF ROBOTIC MANIPULATORS

The optimal control problem in this paper is defined as the trajectory planning of an n degree-of-freedom manipulator from a given initial condition to a target terminal condition in time T such that a prespecified performance index is minimized. Mathematically, this problem can be stated as the minimization of the performance index

$$L = \int_0^T f(\theta(t),\dot{\theta}(t),\mathbf{u}(t),t)\, dt \tag{1}$$

subject to the equation of motion

$$\mathbf{H}(\theta)\ddot{\theta} + \mathbf{g}(\theta,\dot{\theta}) = \mathbf{u}(t)\,,\quad \theta(0) = \theta_0\,,\quad \dot{\theta}(0) = \dot{\theta}_0 \tag{2}$$

and terminal conditions

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$$\theta(T) = \theta_T, \quad \dot{\theta}(T) = \dot{\theta}_T \tag{3}$$

where θ is an $n \times 1$ vector of joint displacements, $\dot{\theta}$ is an $n \times 1$ vector of joint velocities, $\mathbf{u}$ is an $n \times 1$ vector of control variables and T (italics) represents terminal time. A state vector $\mathbf{x}$ is defined as

$$\mathbf{x}(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \tag{4}$$

For sufficiently small variations $\delta\mathbf{x}$ and $\delta\mathbf{u}$ from a nominal trajectory, the performance index can be expanded to second order terms as

$$L(\mathbf{x}+\delta\mathbf{x},\mathbf{u}+\delta\mathbf{u}) \cong L(\mathbf{x},\mathbf{u}) + \int_0^T [f_\mathbf{x}^T\delta\mathbf{x} + f_\mathbf{u}^T\delta\mathbf{u} + \frac{1}{2}(\delta\mathbf{x}^T f_{\mathbf{xx}}\delta\mathbf{x} + \delta\mathbf{u}^T f_{\mathbf{uu}}\delta\mathbf{u}) + \delta\mathbf{x}^T f_{\mathbf{xu}}\delta\mathbf{u}\,]\,dt \tag{5}$$

where $\mathbf{x}$ satisfies requirements on the boundaries, *i.e.*,

$$\mathbf{x}(0) = \begin{bmatrix} \theta(0) \\ \dot{\theta}(0) \end{bmatrix}, \quad \mathbf{x}(T) = \begin{bmatrix} \theta(T) \\ \dot{\theta}(T) \end{bmatrix} \tag{6),(7}$$

In Eq. (5) superscript T denotes transpose. The equation of motion, *i.e.*, Eq. (2), can be expanded to first order about the nominal trajectory giving

$$\mathbf{H}_d\delta\ddot{\theta} + \left(\frac{\partial\mathbf{g}}{\partial\dot{\theta}}\right)_d\delta\dot{\theta} + \left(\ddot{\theta}\frac{\partial\mathbf{H}}{\partial\theta} + \frac{\partial\mathbf{g}}{\partial\theta}\right)_d\delta\theta = \delta\mathbf{u} \tag{8}$$

Subscript d signifies that the terms are evaluated on the nominal trajectory.

The approximate performance index given in Eq. (5) can be minimized by solving the following the LQ problem: Minimize

$$\delta L(\delta\mathbf{x},\delta\mathbf{u}) = \int_0^T [f_\mathbf{x}^T\delta\mathbf{x}+f_\mathbf{u}^T\delta\mathbf{u}+\frac{1}{2}(\delta\mathbf{x}^T f_{\mathbf{xx}}\delta\mathbf{x}+\delta\mathbf{u}^T f_{\mathbf{uu}}\delta\mathbf{u})+\delta\mathbf{x}^T f_{\mathbf{xu}}\delta\mathbf{u}]\,dt \tag{9}$$

subject to

$$\mathbf{M}\delta\ddot{\theta} + \mathbf{C}\delta\dot{\theta} + \mathbf{K}\delta\theta = \delta\mathbf{u}\ , \quad \delta\theta(0) = \delta\dot{\theta}(0) = 0 \tag{10}$$

and fixed values of state variations at the terminal time

$$\delta\theta(T) = \delta\dot{\theta}(T) = 0 \tag{11}$$

where

$$\mathbf{M} = \mathbf{H}_d,\ \mathbf{C} = \left(\frac{\partial\mathbf{g}}{\partial\dot{\theta}}\right)_d,\ \mathbf{K} = \left(\ddot{\theta}\frac{\partial\mathbf{H}}{\partial\theta} + \frac{\partial\mathbf{g}}{\partial\theta}\right)_d \tag{12)-(14}$$

The following section develops an efficient procedure for solving this LQ problem. Once solved, the joint coordinate variables are updated according to the following rules:

$$\theta + \delta\theta \rightarrow \theta,\ \dot{\theta} + \delta\dot{\theta} \rightarrow \dot{\theta},\ \ddot{\theta} + \delta\ddot{\theta} \rightarrow \ddot{\theta} \tag{15)-(17}$$

The control vector is updated using

$$\mathbf{H}(\theta+\delta\theta)[\,\ddot{\theta}+\delta\ddot{\theta}\,] + \mathbf{g}(\theta+\delta\theta,\dot{\theta}+\delta\dot{\theta}) \rightarrow \mathbf{u} \tag{18}$$

(instead of using

$$\mathbf{u} + \delta\mathbf{u} \rightarrow \mathbf{u} \tag{19}$$

adopted in [30]). These update laws guarantee the satisfaction of the boundary conditions as well as the functional relationship between the state and control vectors defined by the equation of motion. The iterative process is repeated until convergence. In summary, the quasi-linearization approach converts a nonlinear optimal control problem into a sequence of LQ problems.

STATE PARAMETERIZATION FOR LQ PROBLEMS

Eqs. (9)-(11) define a time-varying LQ problem with fixed terminal states. Since this problem must be solved in every iteration of the quasi-linearization, a robust and efficient solution approach is essential. Due to the fixed terminal value of the state vector, a solution procedure of the LQ problem based upon the Riccati equation is not applicable. An approach based on calculus of variations is possible [31]. However, such an approach requires the integration of several additional differential equations and suffers from numerical instability problems caused by infinite gains at the terminal time.

In order to provide an efficient algorithm, Bashein and Enns [30] developed a control parameterization approach that converts an LQ problem into a quadratic programming problem. In this approach, the trajectory is characterized by values of the control variables at a finite number of time intervals. The differential constraints imposed by the state equations are then converted into a system of algebraic equality constraints where a state transition matrix is used to determine the linear dependencies between state and control variables. A similar discrete approximation for continuous-time state equations is available [32].

In this study, a version of the state parameterization approach [13] modified to account for problems with fixed terminal states is adopted. Compared to control parameterization [30], this approach reduces the dimensionality of the unknown variables. The basic idea of state parameterization is to represent each generalized coordinate (in this case, the variations of the joint displacements) by a set of known functions φ_j with unknown weighting coefficients e_{ij} (*i.e.*, state parameters) such that

$$\delta\theta_i(t) = \sum_{j=1}^{J} e_{ij}\,\varphi_j(t) \tag{20}$$

subject to boundary conditions, such as Eqs. (10) and (11). Since relationships among the coefficients can be established from the boundary conditions, the number of unknown variables can be reduced.

In particular, in Fourier-based state parameterization each joint coordinate variation is represented by the sum of a fifth-order polynomial and a Fourier-type series:

$$\delta\theta_i(t) = D_i(t) + F_i(t) \tag{21}$$

$$D_i(t) = \sum_{j=0}^{5} d_{ij}\,t^j \tag{22}$$

$$F_i(t) = \sum_{j=1}^{J} (\ a_{ij}\cos\frac{2j\pi t}{T} + b_{ij}\sin\frac{2j\pi t}{T}\ ) \tag{23}$$

This approach characterizes the joint coordinate variations by boundary values and their first and second derivatives (*i.e.*, $\delta\theta_i$, $\delta\dot{\theta}_i$, and $\delta\ddot{\theta}_i$ at both boundaries) and the coefficients of the Fourier-type series. That is, the following equations can be formulated:

$$\delta\theta_i(0) = \delta\theta_{io} = D_i(0) + F_i(0),\ \delta\theta_i(T) = \delta\theta_{iT} = D_i(T) + F_i(T) \tag{24),(25}$$

$$\delta\dot{\theta}_i(0) = \delta\dot{\theta}_{io} = \dot{D}_i(0) + \dot{F}_i(0),\ \delta\dot{\theta}_i(T) = \delta\dot{\theta}_{iT} = \dot{D}_i(T) + \dot{F}_i(T) \tag{26),(27}$$

$$\delta\ddot{\theta}_i(0) = \delta\ddot{\theta}_{io} = \ddot{D}_i(0) + \ddot{F}_i(0),\ \delta\ddot{\theta}_i(T) = \delta\ddot{\theta}_{iT} = \ddot{D}_i(T) + \ddot{F}_i(T) \tag{28),(29}$$

from which the coefficients of the polynomials can be determined as functions of boundary values and Fourier-type coefficients [15].

Using this idea of state parameterization, a joint coordinate variation vector can be written as:

$$\delta\theta = \rho y + p \tag{30}$$

where

$$\delta\theta = [\ \delta\theta_1\ \ \delta\theta_2\ \cdots\ \delta\theta_n\ ]^T \tag{31}$$

$$y = [\ y_1^T\ \ y_2^T\ \cdots\ y_n^T\ ]^T \tag{32}$$

$$p = [\ p_1\ \ p_2 \cdots\ p_n\ ]^T \tag{33}$$

with

$$y_i = [\ \delta\theta_{i0}\ \ \delta\theta_{iT}\ \ a_{i1} \cdots\ a_{iJ}\ \ b_{i1} \cdots b_{iJ}\ ]^T \tag{34}$$

$$p_i = p_i(\delta\theta_{i0}, \delta\dot{\theta}_{i0}, \delta\theta_{iT}, \delta\dot{\theta}_{iT}, t) \tag{35}$$

In the above equations y_i is a state parameter vector of $\delta\theta_i$ and consists of unknown boundary values and coefficients of Fourier-type series of the i-th joint coordinate. **p** is a vector of time and known boundary values of the joint coordinates. ρ represents a matrix whose elements are known functions of time. Detailed derivations are can be found in [13].

Direct differentiation of Eq. (30) gives

$$\delta\theta = \sigma y + q,\ \ \delta\dot{\theta} = \phi y + r \tag{36),(37}$$

where

$$\sigma = \dot{\rho},\ \ q = \dot{p},\ \ \phi = \ddot{\rho},\ \ r = \ddot{p} \tag{38}$$

To apply this state parameterization approach to solve the LQ problem defined by Eqs. (9)-(11), Eqs. (30), (37), and (38) are first substituted into the linearized equation of motion, Eqs. (10). A control variation vector can thus be written as a function of y and time. This control variation vector and equations (30), (37), and (38) can be substituted into the performance index, Eq. (9). Hence, the performance index can be written as a function of y

$$\delta L = y^T \Delta y + y^T \Gamma + \Sigma \tag{39}$$

where the capital Δ, Γ, and Σ are functions of time and known boundary values. The necessary condition for minimum performance index is given by

$$\frac{d\,(\delta L)}{d\,y} = 0 \tag{40}$$

which is equivalent to

$$(\ \Delta + \Delta^T\ )\ y = -\Gamma \tag{41}$$

Eq. (48) represents a system of linear algebraic equations that can be solved for the optimal values of the state parameter vector.

This Fourier-based state parameterization approach [13] converts the original optimal control problem into an algebraic optimization problem, which is typically of lower order than that obtained via control parameterization. Satisfactory near optimal results can often be obtained with only a few term Fourier-type series.

SIMULATION STUDIES

Example 1

This example considers the minimum energy trajectory of a one-link manipulator. The problem is to find the trajectory that minimizes the performance index

$$L = \int_0^1 u^2\,dt \tag{42}$$

subject to

$$\ddot{\theta} + g \sin\theta = u(t) \qquad \theta(0) = \theta_0,\ \ \dot{\theta}(0) = \dot{\theta}_0 \tag{43}$$

with final states specified as:

$$\theta(1) = 1 \text{ rad},\ \ \dot{\theta}(1) = 0 \text{ rad/s} \tag{44}$$

where Eq. (43) is the equation of motion of the manipulator with g as the gravitational constant (9.805 m/s2).

Computer programs were written in the "C" language and compiled by the Microsoft Quick C compiler (Version 1.0). The programs were executed on a 16 MHz NEC 386 PowerMate computer with a 16 MHz co-processor.

A three-term Fourier-type series was used for the state parameterization. The initial nominal trajectory was assumed to be a fifth order polynomial that satisfies the boundary conditions: joint displacements and velocities (given at both ends) and joint accelerations (assumed zero at both ends.) The value of the performance index, plotted in Figure 1, converged after seven iterations, although the minimum is closely approximated after only one iteration. The total execution time was 2.6 s.

To verify the results, this problem was solved using a numerical method based upon calculus of variations. A two-point boundary-value problem was formulated using the necessary conditions of optimality, and then a steepest-descent method [33] was implemented. In order to apply the steepest-descent algorithm, the problem was converted to a free terminal state problem with a modified performance index:

$$\bar{L} = 20000(\theta(1) - 1)^2 + 200\dot{\theta}^2(1) + \int_0^T u^2\,dt \tag{45}$$

where the weightings on the terminal states were determined by trial and error to ensure a terminal state error of less than 10^{-3} rad. In implementing the steepest-descent method the two-point boundary-value problem was integrated by a fourth-order Runge-Kutta routine with a time step of 0.01 s. The computer program terminated execution once the difference between two subsequent values of the performance index was less than 10^{-6} N^2. The initial guess was identical to the one used in the quasi-linearization approach. Under such circumstances, a solution was obtained after 4725 iterations. The execution time was 2060 s, almost three orders of magnitude longer than the proposed approach.

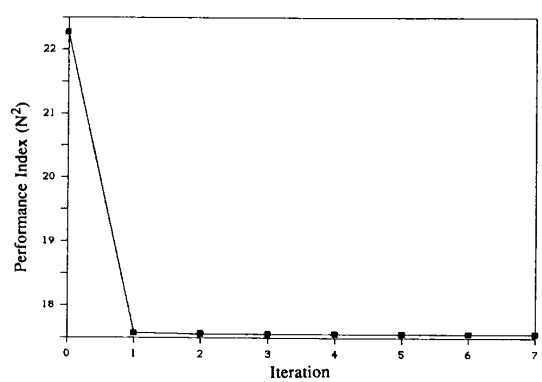


Figure 1. Value of Performance Index as a Function of Iteration for Example 1.

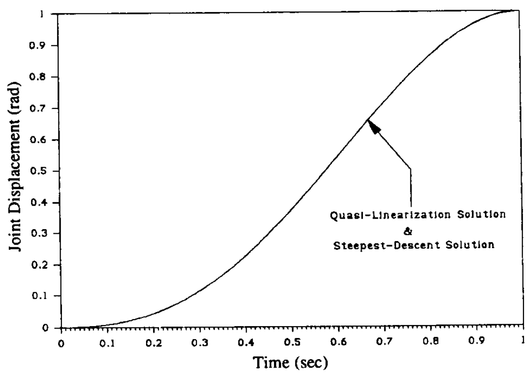


Figure 2. Joint Displacement History for Example 1.

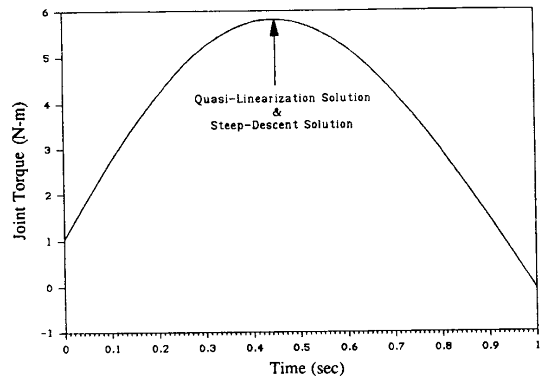


Figure 3. Joint Torque History for Example 1

The joint displacements of both approaches are plotted in Figure 2. The time histories of the control variables are given in Figure 3. In both figures, the solutions of the two approaches are indistinguishable. The approach of quasi-linearization, with state parameterization for solving the LQ problem, offers significant savings in computational cost with comparable accuracy.

Example 2

This example considers a minimum energy problem of a two-link planar manipulator where the integrand of the performance index is specified as the sum of the joint torques squared. A schematic diagram of the manipulator is given in Figure 4. The boundary conditions were chosen to be

$$\theta_1(0) = 0 \text{ rad},\ \dot{\theta}_1(0) = 0 \text{ rad/s} \quad (46)$$

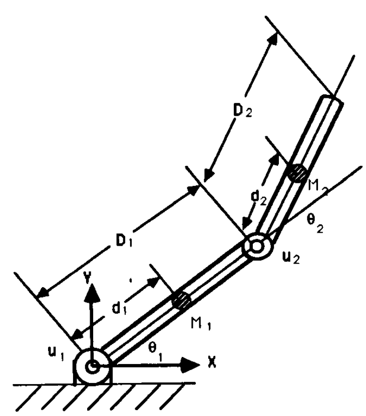
$$\theta_2(0) = 1 \text{ rad},\ \dot{\theta}_2(0) = 0 \text{ rad/s} \quad (47)$$

$$\theta_1(1) = -2.6223 \text{ rad},\ \dot{\theta}_1(1) = -5.8314 \text{ rad/s} \quad (48)$$

$$\theta_2(1) = -0.0134 \text{ rad},\ \dot{\theta}_2(1) = 10.284 \text{ rad/s} \quad (49)$$

where the terminal boundary conditions were selected in such a way that the optimal solution is torque free motion. Using a three-term Fourier-type series, the history of the performance index during the optimization process is given in Figure 5. The near optimal solution is closely approximated in five iterations. Rapid convergence is again observed. The corresponding value of the performance index is 3.425 N^2-m^2. The near optimal and "true" optimal joint displacements are plotted in Figures 6. The control vector is plotted in Figure 7.

In Figure 7, the oscillatory behavior of the control variable solutions is due to the existence of the Fourier-type series used in the state parameterization. The accuracy can be improved by increasing the number of terms of the Fourier-type series. For example, the performance index is reduced to 1.492 N^2-m^2 when using a five term Fourier-type series.



$M_1 = M_2 = 1.0$kg $I_1 = I_2 = 0.1$kg-m^2
$D_1 = D_2 = 1.0$m $d_1 = d_2 = 0.5$m

Figure 4. Schematic Diagram of Manipulator for Example 2.

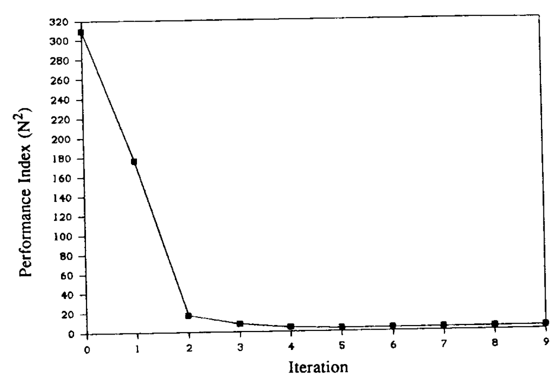


Figure 5. Value of Performance Index as a Function of Iteration for Example 2.

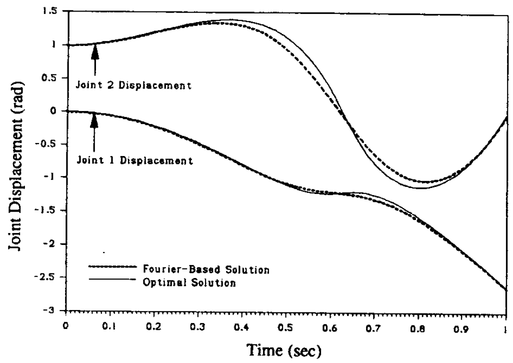


Figure 6. Joint Displacement Histories for Example 2.

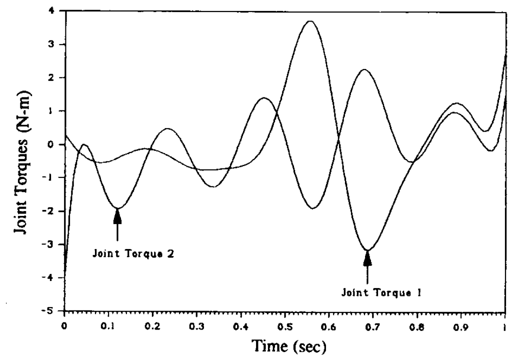


Figure 7. Joint Torque Histories for Example 2 (Fourier-Based Solution).

SUMMARY

In this paper, a numerical algorithm has been developed to solve the problem of optimal control of robotic manipulators. A quasi-linearization method is used to convert a nonlinear optimal control problem into a sequence of LQ problems which are solved by an efficient state parameterization approach. The update laws for the nominal trajectory ensure satisfaction of the terminal conditions.

In contrast to dynamic-programming-based methods, the proposed approach does not demand extensive computer storage requirements and thus is capable of achieving optimality without limiting the degrees-of-freedom of the trajectory. Compared to nonlinear-programming-based methods, the approach offers significant advantages in computational efficiency. Compared to calculus-of-variations-based methods, the approach eliminates the requirement of solving a two-point boundary-value problem and therefore is more robust and efficient. Work is currently underway to extend this method to problems with constrained state and control variables and to high order robotic manipulator problems.

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