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Technical Briefs

A Geometric Representation of Root Sensitivity

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In this paper, we present a geometric method for representing the classical root sensitivity function of linear time-invariant dynamic systems. The method employs specialized eigenvalue plots that expand the information presented in the root locus plot in a manner that permits determination by inspection of both the real and imaginary components of the root sensitivity function. We observe relationships between root sensitivity and eigenvalue geometry that do not appear to be reported in the literature and hold important implications for control system design and analysis.

Introduction

In classical control theory the root sensitivity, S_p, is defined as the relative change in a system root or eigenvalue, λ_i ($i=1, \ldots, n$), with respect to a system parameter, p. Most often, the parameter analyzed is the forward loop gain, k. The root sensitivity with respect to gain is given by

$$S_k=\frac{d\lambda(k)/\lambda(k)}{dk/k}=\frac{d\lambda(k)}{dk}\,\frac{k}{\lambda(k)} \tag{1}$$

Since the eigenvalues may occur as complex conjugate pairs, S_k may be complex.

Equation (1) is often introduced in determining the break points of the Evans root locus plot for single-input single-output systems. At the break points, S_k becomes infinite as at least two of the n system eigenvalues undergo a transition from the real domain to the complex domain or vice versa. This transition causes an abrupt change in the relation between the eigenvalue angle $\angle\lambda$ and gain k yielding an infinite eigenvalue derivative $d\lambda/dk$ (Ogata, 1990).

The root sensitivity function S_k is a measure of the effect of parameter variations on the eigenvalues. S_k is an important quantity in light of a key objective of feedback control theory, the reduction of the system sensitivity to variations in system parameters. For example, the control system of a robotic manipulator should be relatively insensitive to the payload carried by the arm for the recommended payload range. If the manipulator's performance is sensitive to payload variations, then the control system is not robust and performance is difficult to guarantee. In this case, S_m, where m is the payload mass, should be relatively small over the manipulator's payload mass range. Such considerations are critical if control designers are to develop high performance, robust, closed-loop systems.

In this paper, we present a geometric technique for determining root sensitivity. The technique relies on a set of gain plots (Kurfess and Nagurka, 1991) that are an alternate visualization of the Evans root locus plot. In particular, we prove that the slopes of the gain plots are directly related to the real and imaginary components of the sensitivity function. An example is presented demonstrating the insight gained via this geometric perspective on the sensitivity function, and its utility in control system design.

Root Sensitivity Analysis

In this section, we derive an expression for the complex root sensitivity function by employing a polar representation of the eigenvalues in the complex plane. We posit three assumptions: (i) the systems analyzed are lumped parameter, linear time-invariant (LTI) systems; (ii) there are no eigenvalues at the origin of the s-plane, i.e.,

$$\lambda_i\neq 0,\quad \forall i=1,\ldots,n \tag{2}$$

(although the eigenvalues may be arbitrarily close to the origin singularity); and (iii) the forward scalar gain, k, is real positive, i.e., $k\in\mathbb{R}$, $k>0$. Based on these assumptions, we draw the following observations: the real component of the sensitivity function is given by

$$\mathrm{Re}\{S_k\}=\frac{d\ln|\lambda(k)|}{d\ln(k)} \tag{3}$$

and the imaginary component of the sensitivity function is given by

$$\mathrm{Im}\{S_k\}=\frac{d\angle\lambda(k)}{d\ln(k)} \tag{4}$$

These observations may be proven as follows. Equation (1) may be rewritten (Horowitz, 1963; Kuo, 1991) in terms of the derivatives of natural logarithms as

$$S_k=\frac{d\ln(\lambda(k))}{d\ln(k)} \tag{5}$$

The natural logarithm of the complex value, λ, is equal to the sum of the logarithm of the magnitude of λ and the angle of λ multiplied by $j=\sqrt{-1}$. Thus, Eq. (5) becomes

$$S_k=\frac{d[\ln|\lambda(k)|+j\angle\lambda(k)]}{d\ln(k)} \tag{6}$$

Since j is a constant, Eq. (6) may be rewritten as

$$S_k=\frac{d\ln|\lambda(\mathrm{k})|}{\mathrm{d}\ln(k)}+j\,\frac{d\angle\lambda(k)}{d\ln(k)} \tag{7}$$

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The complex root sensitivity function is now expressed with distinct real and imaginary components employing the polar form of the eigenvalues. It follows from assumption (ii) that $\ln(k)$ is real. (In general, most parameters studied are real and this proof is sufficient. If, however, the parameter analyzed is complex as explored in Nagurka and Kurfess (1992), it is a straightforward task to extend the above analysis.)

The proof is completed by taking the real and imaginary components of Eq. (7), yielding Eqs. (3) and (4). It is interesting to note that the Cartesian representation of S_k is related to the polar representation of λ.

Geometric Relations to Gain Plots

The gain plots (Kurfess and Nagurka, 1991) are an alternate graphical representation of the Evans root locus plot. They explicitly graph the eigenvalue magnitude versus gain in a magnitude gain plot and the eigenvalue angle versus gain in an angle gain plot. The magnitude gain plot employs a log-log scale whereas the angle gain plot uses a semi-log scale (with the logarithms being base 10.) Although gain is the variable of interest, any parameter may be used in the geometric analysis.

We next make the observation that the slope of the magnitude gain plot is the real component of S_k. The magnitude gain plot slope, M_m, is

$$M_m = \frac{d\log(|\lambda(k)|)}{d\log(k)} \tag{8}$$

which may be rewritten as

$$M_m = \frac{d[\log(e)\ln(|\lambda(k)|)]}{d[\log(e)\ln(k)]} = \frac{d\ln(|\lambda(k)|)}{d\ln(k)} \tag{9}$$

corresponding to Eq. (3).

Furthermore, the slope of the angle gain plot is the product of the imaginary component of S_k and the constant, $(\log(e))^{-1}$. The angle gain plot slope, M_a, is

$$M_a = \frac{d\angle\lambda(k)}{d\log(k)} \tag{10}$$

which may be rewritten as

$$M_a = \frac{d\angle\lambda(k)}{d[\log(e)\ln(k)]} = \frac{1}{\log(e)}\frac{d\angle\lambda(k)}{d\ln(k)} \tag{11}$$

and hence M_a is proportionally related to Eq. (4) by $(\log(e))^{-1}$.

Example

In this section we demonstrate the graphical method of determining the root sensitivity function. The example involves a system using positive modal feedback (Sardar and Paul, 1991). An example of a PD controller has been reported previously (Kurfess and Nagurka, 1992). Here a more complicated system is used to demonstrate the determination of the root sensitivity function with respect to a system parameter, namely stiffness, and its implictions for design.

Positive Modal Feedback Example. Sardar and Paul (1991) considered the application of positive modal feedback to the dynamic system, shown in Fig. 1, representing a lumped parameter structural model. The block diagram of the positive model feedback is shown in Fig. 2. The transfer function between the actual and desired position is

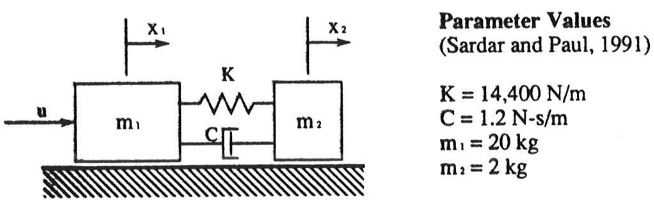

Fig. 1 Lumped parameter system model

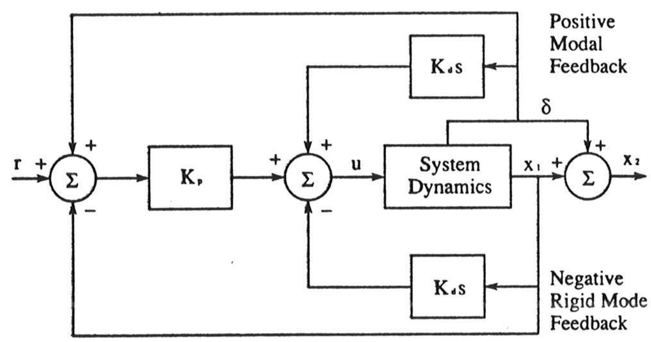

Fig. 2 Block diagram with PD control and positive modal feedback

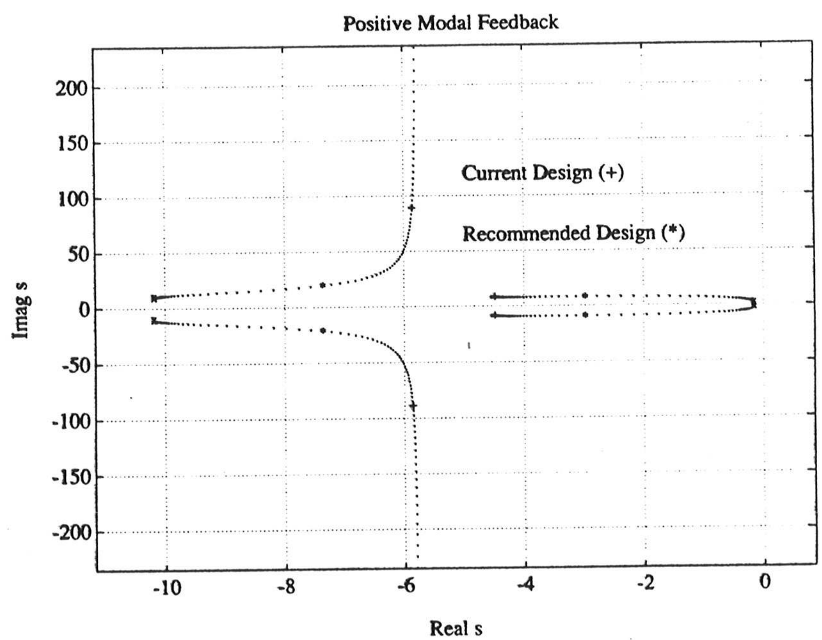

Fig. 3 Root locus plot for Eq. (12)

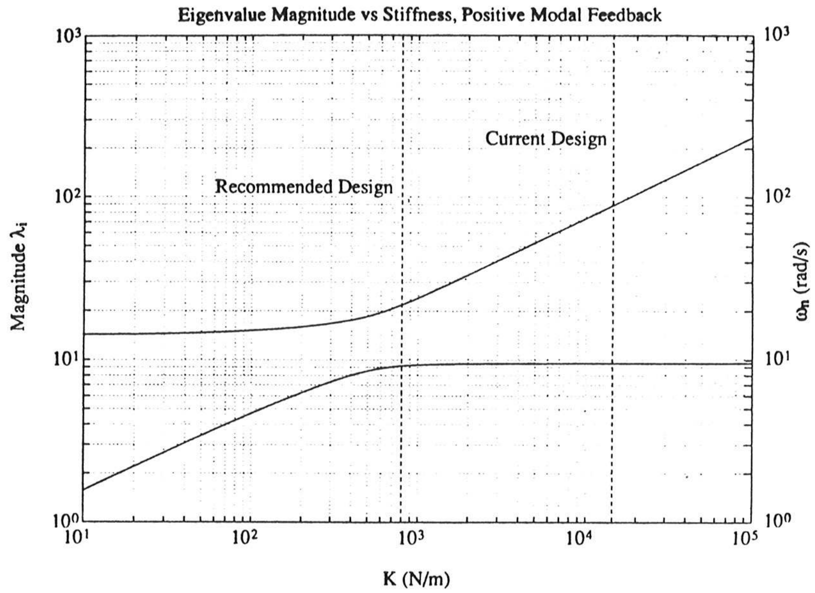

Fig. 4(a) Magnitude gain plot for Eq. (12)

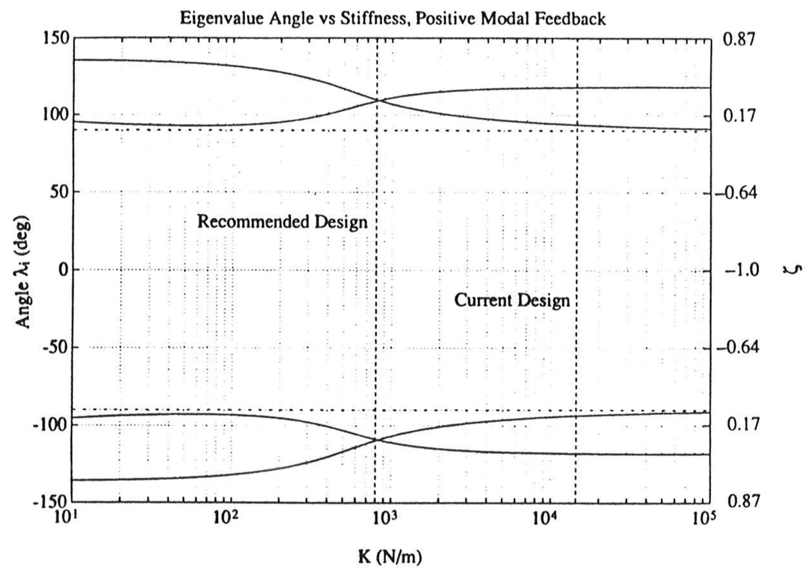

Fig. 4(b) Angle gain plot for Eq. (12)

$$\frac{X_2(s)}{R(s)}=\frac{K_p(Cs+K)}{b_4s^4+b_3s^3+b_2s^2+b_1s+b_0} \tag{12}$$

where

$$b_4=m_1m_2 \tag{13}$$

$$b_3=2m_2K_d+(m_1+m_2)C \tag{14}$$

$$b_2=2m_2K_p+(m_1+m_2)K+K_dC \tag{15}$$

$$b_1=K_pC+K_dK \tag{16}$$

$$b_0=K_pK \tag{17}$$

where K_p is the proportional gain and K_d is the derivative gain. The term, δ, in Fig. 2 is the relative displacement of the masses given by

$$\delta=x_2-x_1 \tag{18}$$

In their work, Sardar and Paul assumed a value of $K_d=200$ N-s/m, and varied the value of K_p to generate root locus plots. It is our objective to pick a value of K_p and then conduct a sensitivity analysis of the closed-loop system with respect to the spring stiffness, K. We use a value of $K_p=2000$ N/m, yielding a stable system, and sweep K through a large range (including the value used by Sardar and Paul, $K=14400$ N/m). Figure 3 is the root locus for the closed-loop system as K is varied in the range $10^1 \leq K \leq 10^5$. (Note that the root locus axes are *not* square.) Figures 4(a,b) and 5(a, b) are the gain plots and sensitivity plots, respectively, for the system. In the gain plots, natural frequency and damping ratio vertical axes have been added to aid in the analysis.

From these figures, several interesting features are available. First, the entire range of K corresponds to a closed-loop stable system. The design does, however, place a set of high frequency poles near the imaginary axis. These poles represent the mode of the system. The designer cannot ignore these poles since they do not decay fast enough, with respect to the other pole pair, to invoke the dominant pole theory. The design may meet specifications; however, let us assume that we would like more damping. There is the possibility of reducing the stiffness, K, which results in increasing the damping, ζ. By inspection from the angle gain plot, the value of K that results in the highest damping for both pole pairs is approximately $K=870$ N/m. This is where the damping ratio for both pole pairs is approximately equal. At $K=870$ N/m, we can read the natural

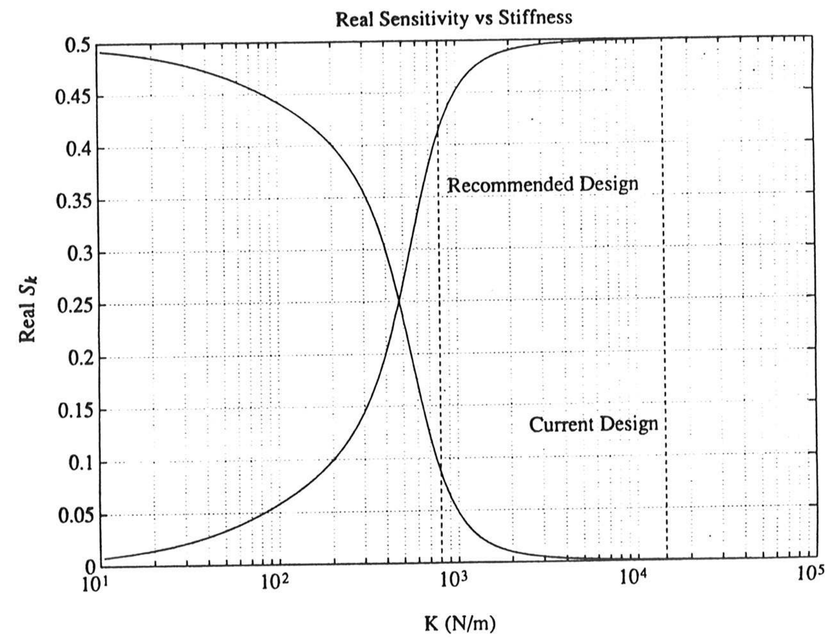

Fig. 5(*a*) Real component of S_k for Eq. (12)

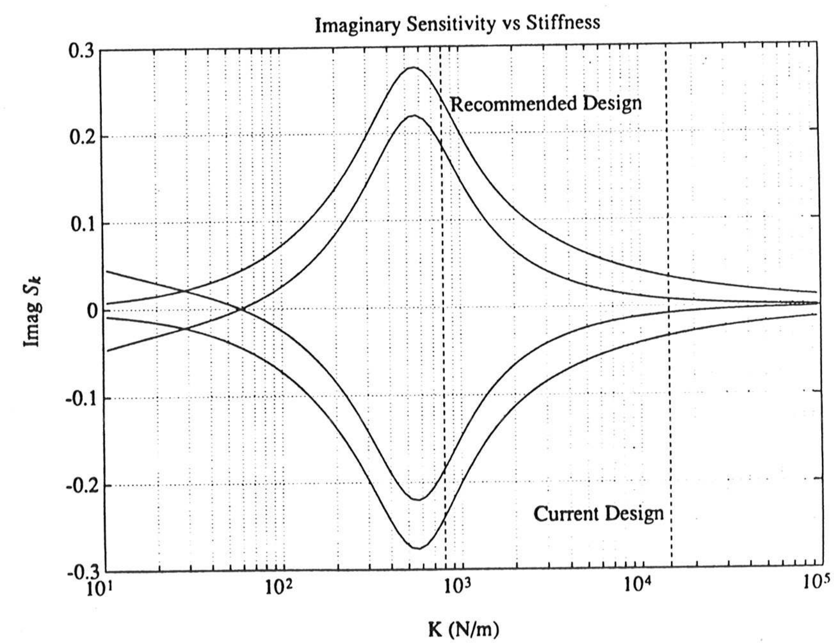

Fig. 5(*b*) Imaginary component of S_k for Eq. (12)

frequency for each pole pair from the magnitude gain plot to be 9 and 21 rad/s, respectively.

We may now examine the sensitivity plots to see if our design is robust to variations in K. The plots suggest that the design is fairly robust to stiffness variations. The real part of the sensitivity function indicates sensitivities of 0.07 and 0.42 for the lower and higher frequency pole pairs, respectively. In other words a 10 percent increase in K yields a 0.7 percent increase in the low frequency pole pair's natural frequency and a 4.2 percent variation in the natural frequency of the high frequency poles. The values of the imaginary part of the root sensitivity function are approximately 0.24 and 0.18 for the higher and lower frequency pole pairs, respectively. Again, this translates directly to the rate at which the pole angles (and thus damping) change as a function of K.

From Figs. 5(*a*, *b*), the control designer may observe that the recommended design does not generate a highly sensitive system. Furthermore, higher stiffnesses relate to smaller imaginary components of the sensitivity function. For higher stiffnesses, the real part of the root sensitivity for the higher frequency pole pair increases asymptotically to 0.5, while the lower frequency pair approaches zero. Several design insights may be garnered from these plots. First, a stiffness of approximately 500 N/m may not be desirable if low damping variations are critical, since the corresponding imaginary part of the sensitivity function is maximum. However, at a K of approximately 400 N/m, the real part of the root sensitivity function is equal for both pole pairs. Therefore, if the radial pole motion of both pairs is to be minimized, a stiffness of 400 N/m should be considered. Finally at high stiffnesses such as 10^4 N/m, the root sensitivity function is relatively flat. Thus, stiffness variations at these higher values result in predictable changes in pole locations.

Closing

The concept of root sensitivity in classical controls is often introduced to emphasize the high "sensitivity" of eigenvalues with respect to a system parameter such as gain near the break points. Normally, the root sensitivity function is not discussed as a complex quantity in control system analysis and design. Here, we have derived and demonstrated a powerful yet uncomplicated means of visualizing the root sensitivity function. The slopes of the gain plots provide a direct measure of the real and imaginary components of the root sensitivity, and are available by inspection. The use of the gain plots in conjunction with other traditional graphical techniques offers the control system designer important information for selection of appropriate system parameters.

Acknowledgments

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Synthesis of Multivariable Controller With Plant Test Data

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A method to synthesize decoupled multivariable control system from a batch of plant test data is introduced. The method is applicable when the system has more inputs than outputs and is open-loop stable. An advantage of this method is that explicit identification of an open-loop system model is not required for controller synthesis.

I Introduction

In many multivariable control system syntheses, the objectives of the design are 1) to have each output of the system meet desired command response characteristics, and 2) to properly suppress the disturbance response. Normally, the first objective dictates a decoupled output response and is achieved if the input-output transfer function matches some predetermined Laplace domain rational function; the second objective is achieved if a disturbance-attenuation filtering function can be placed in the system equation. Often, the system under consideration is open-loop stable.

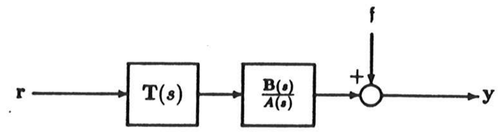

Fig. 1 Basic system configuration with feedforward cascaded controller

For the controller design, precise plant models are generally needed. In many cases, such models are not available. The plant, however, may be available for testing. Controller design techniques based on plant test data had thus been developed (Aström, 1980; Owens, 1984; Rogers and Owens, 1990). However, these techniques still require approximated plant models, so that the control system design still consists of two separate steps, namely, plant model estimation and controller synthesis. In this paper, we present a method to synthesize the controller directly from plant test data. This, in effect, combines the estimation and synthesis procedures into one design process, thereby simplifying the task.

II Statement of the Problem

In the present study, we will concentrate on the following problem: Given an overall stable system with Laplace-domain equation (Maciejowsky, 1989)

$$\mathbf{y}(s)=\frac{\mathbf{B}(s)}{A(s)}\mathbf{u}(s)+\mathfrak{f}(s) \tag{1}$$

where $\mathbf{y}$ is an $p_o\times 1$ output vector, $\mathbf{u}$ is an $p_i\times 1$ input vector, $\mathfrak{f}$ is an $p_o\times 1$ disturbance vector, $A(s)$ is an nth order polynomial and $\mathbf{B}(s)$ is an $p_o\times p_i$ matrix of $\bar{n}$th order polynomials, design a controller so that the system outputs meet the response characteristics specified by the rational function $\mathfrak{h}(s)=[N(s)/D(s)]$ with given polynomials $D(s)$ and $N(s)$.[2]

In theory, an open-loop cascaded controller, $\mathbf{T}(s)$ (see Fig. 1) will serve the purpose if $[\mathbf{B}(s)/A(s)]\,\mathbf{T}(s)=\mathfrak{h}(s)\mathbf{I}$. In practice, however, several major difficulties may arise:

(a) $\mathbf{T}(s)$ will not be implementable if $\mathbf{B}(s)/A(s)$ does not have a stable inverse when $p_i=p_o$;

(b) $\mathbf{T}(s)$ is an open-loop compensator and is thus sensitive to disturbance;

(c) An estimate of $\mathbf{B}(s)/A(s)$ may be not available so that $\mathbf{T}(s)$ cannot be computed.

The first problem is discussed in the Appendix, where it is shown that if $p_i=p_o+1$ and $\mathbf{B}(s)/A(s)$ is of full row rank for all s, then there always exists, for some $m\geq l$, an appropriate polynomial matrix

$$\mathbf{R}(s)=\sum_{l=0}^{m}\mathcal{R}_l s^{m-l}$$

where $\mathcal{R}_l$ are constant $p_i\times p_o$ matrices which will satisfy the relation

$$\frac{\mathbf{B}(s)}{A(s)}\,\frac{\mathbf{R}(s)}{s^l\mathcal{G}(s)}=\frac{1}{s^l}\mathbf{I} \tag{2}$$

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[2] (a) The value $l=n-\bar{n}$ signifies the relative order of $\mathbf{B}(s)/A(s)$. For causal systems, we will have $1\leq l\leq n$. (b) If we denote the order of $D(s)$ as n_D and that of $N(s)$ as n_N, then the causality of the formulation demands that $n_D\geq l+n_N$.